# Interest Rate Modelling

P. Ouwehand

Department of Mathematical Sciences
Stellenbosch University

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# 1 Change of Numéraire

## 1.1 Stochastic Financial Modelling

Recall the following set–up:

- A market model is a tuple

$$\mathcal{M} = (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (S_t^0, \ldots, S_t^N)_{t\geq 0}$$

  where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\mathcal{F}_t)_t$ a filtration satisfying the usaul conditions, and $S_t = (S_t^0, \ldots S_t^N)$ an $(N+1)$–dimensional adapted cádlág semimartingale.

- We will often assume a finite horizon $[0, T]$, e.g. to price European options.

- We also make the usual assumptions about the market:

  - No transaction costs
  - Continuous trading
  - Liquid markets for every security
  - Short sales allowed
  - Perfect divisibility of assets

To get results, we will usually specialize: We will generally assume that $\Omega$ comes with a $K$–dimensional Brownian motion $W_t = (W_t^1, \ldots, W_t^K)$ which generates the filtration $\mathcal{F}_t$ (augmented). We say that we have $K$ *sources of noise*. Further, we assume that the asset price process is given by an Ito diffusion:

$$dS_t = \mu(t, S_t)\; dt + \sigma(t, S_t)\; dW_t$$

which is shorthand for

$$d\begin{pmatrix} S_t^0 \\ \vdots \\ S_t^N \end{pmatrix} = \begin{pmatrix} \mu^0(t, S_t) \\ \vdots \\ \mu^N(t, S_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{01}(t, S_t) & \ldots & \sigma_{0K}(t, S_t) \\ \vdots & \vdots & \vdots \\ \sigma_{N1}(t, S_t) & \ldots & \sigma_{NK}(t, S_t) \end{pmatrix} \begin{pmatrix} dW_t^1 \\ \vdots \\ dW_t^K \end{pmatrix}$$

Under these conditions, the asset price process is (strong) Markov.

Generally, we make another assumption on $S_t^0$: we assume that it is the money market account process ("riskless" bank account process, which has dynamics

$$dS_t^0 = rS_t^0\,dt \qquad S_0^0 = 1$$

A *numéraire* is a price process $N_t$ which has $N_t > 0$ a.s. Think of a numéraire as a *unit* into which other assets are translated. Thus if $S_t$ is the price of $S$ in money, then $\hat{S}_t = \frac{S_t}{N_t}$ is the price of $S$ in units of $N$.

We often choose the numéraire to be the money market account process $S_t^0$. In that case, we write $\bar{S}_t = \frac{S_t}{S_t^0}$ for the value of $S_t$ in terms of the numéraire. Of course, $\bar{S}_t$ is just the discounted value of $S$ at time $t$.

A European contingent claim $C$ is an derivative which, at some future time $T$ has a payoff which is a known function of asset prices at time $T$, i.e.

$$C_T = f(S_T)$$

so that $C_T$ is an $\mathcal{F}_T$–measurable random variable. The time $T$ is called the *maturity* or *exercise time* of the claim.

Our central problem is the pricing and hedging of such derivatives. A European claim can be priced by arbitrage methods only if there is a trading strategy which exactly replicates its payoff.

**Definition 1.1** A trading strategy/portfolio is a left–continuous (or, more generally, predictable) process $\phi_t = (\phi_t^0, \ldots, \phi_t^N)$ which is integrable w.r.t. the semimartingale $S_t$.

$\phi_t^n$ is to be thought of as the number of asset $S^n$ held in the portfolio at time $t$. The value of the portfolio at time $t$ is

$$V_t(\phi) = \phi_t \cdot S_t = \sum_{n=0}^{N} \phi_t^n S_t^n$$

□

In the discrete–time setting, we required trading strategies to be self–financing:

$$\phi_{t-\Delta t} \cdot S_t = \phi_t \cdot S_t \qquad \text{or} \qquad S_t \Delta_t \phi = 0$$

where $\phi_t$ is the portfolio held between times $t$ and $t+1$, and $\Delta_t \phi = \phi_t - \phi_{t-\Delta t}$. In continuous–time, if we let $\Delta t \to 0$, this looks like an SDE

$$S_t\,d\phi_t = 0 \tag{*}$$

However, it would be wrong to use $(*)$ as the self–financing condition in continuous–time, because:

(i) Stochastic integrals are to be interpreted in the Ito sense.

(ii) If $H_t$ is left–continuous, then the stochastic integral

$$\int_0^T H_t\ dX_t = \lim_{||P|| \to 0} \sum H_{t_{n-1}}(X_{t_n} - X_{t_{n-1}})$$

is a limit (in probability) of *left–hand* Riemann Stieltjes sums.

(iii) $S_t(\phi_t - \phi_{t-\Delta t})$ looks like a term in a *right–hand* sum.

This problem is fixed rather easily: Add and subtract $S_{t-\Delta t}\Delta_t\phi$ from the left–hand side of $(*)$ to obtain:

$$S_{t-\Delta t}(\phi_t - \phi_{t-\Delta t}) + (S_t - S_{t-\Delta t})(\phi_t - \phi_{t-\Delta t}) = 0$$

In the continuous–time limit, this looks like

$$S_t\ d\phi_t + d[S,\phi]_t \tag{**}$$

because the first term *is* a left–hand sum, and the second term looks like a summand in the covariation process.

Thus we would like to define a trading strategy $\phi$ to be self–financing if $(**)$ holds. This means that the differential of the value process $V_t(\phi)$ is easily calculated. By Ito's formula,

$$dV_t = d(\phi_t \cdot S_t) = \phi_t\ dS_t + S_t\ d\phi_t + d[S,\phi]_t = \phi_t\ dS_t$$

and thus

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u\ dS_u = V_0(\phi) + G_t(\phi)$$

where the gains process $G_t(\phi)$ is defined to be the obvious stochastic integral.

Intuitively, this is *exactly right*: The value of the portfolio at time $t$ equals the initial "stake" plus the gain/loss made by playing the stock market game. Nothing else is added (or taken away).

**Definition 1.2** A trading strategy $\phi$ is self–financing if and only if $G_t(\phi) = \int_0^t \phi_u\ dS_u$, i.e. iff $d(\phi_t \cdot S_t) = \phi_t \cdot\ dS_t$.

□

Actually, the self–financing condition already sneaked into the derivation of the Black–Scholes PDE, as given by (e.g.) Hull. There, we started with a portfolio $\Pi$ long one vall option $C$ and short $n$ shares $S$, so that $\Pi = C - nS$. We then boldly wrote $d\Pi = dC - n\ dS$, and imposed the condition that $\Pi$ be locally riskless (so that all $dW_t$–terms cancel out, to conclude $n = \frac{\partial C}{\partial S}$. But this was dishonest: We should have had $dV = dC - n\ dS - (S\ dn + d[n,S])$, because the delta of the option ($n$) is also changing! However, if we require the hedge to be self–financing, then $S\ dn + d[n,S] = 0$, and no harm is done. The derivation of the PDE can now continue as before.

**Remarks 1.3** In the literature, other conditions are often imposed on trading strategies to ensure that they are sufficiently well–behaved. For example, a self–financing trading strategy is called *tame* if $V_t(\phi) \geq 0$ a.s. It is called *admissible* if its discounted value is a martingale under the EMM. This is important, because even the Black–Scholes model has "doubling" strategies, and is not arbitrage–free if arbitrarily large losses can be sustained. However, we will ignore thes etechnical points in what follows.

□

**Definition 1.4** A (European) contingent claim $C$ is said to *attainable* if and only if there exists a self–financing strategy $\phi_t$ such that $C_T = V_T(\phi)$ (where $T$ is the exercise date of the claim). Then $\phi$ is called a *replicating portfolio* for $C$.

A market model is *complete* if and only if every contingent claim is attainable.

□

**Proposition 1.5** (Numéraire)
*A self–financing portfolio remains self–financing under a change of numéraire.*

This looks totally obvious: After all if we don't add or subtract funds from our portfolio when we reckon in units of money, we don't add or subtract funds if we reckon in units of barrels of oil either. However, our definition of self–financing is that $d(\phi_t \cdot S_t) = \phi_t \cdot\ dS_t$. Now suppose that we reckon in terms of a new numéraire $N_t$. Let $\hat{S}_t = \frac{S_t}{N_t}$ be the price of $S$ in units of $N$. To prove that the self–financing condition holds, we must show that $d(\phi_t \cdot \hat{S}_t) = \phi_t \cdot\ d\hat{S}_t$, and this no longer seems so obvious.

**Proof:** Let $\hat{V}_t = \frac{V_t(\phi)}{N_t}$. Then by Ito's formula

$$\begin{aligned} d\hat{V}_t &= \frac{1}{N_t}\ dV_t + V_t\ d\left(\frac{1}{N_t}\right) + d[V, \frac{1}{N}]_t \\ &= \frac{\phi_t}{N_t} \cdot dS_t + \phi_t S_t\ d\left(\frac{1}{N_t}\right) + \phi_t \cdot d[S, \frac{1}{N}] \end{aligned}$$

because (by the self–financing condition) $dV_t = \phi_t\ dS_t$, so $d[V, \frac{1}{N}]_t = \phi_t \cdot [S, \frac{1}{N}]_t$. Thus

$$\begin{aligned} d\hat{V}_t &= \phi_t \left(\frac{1}{N_t} \cdot dS_t + S_t\ d\left(\frac{1}{N_t}\right) + d[S, \frac{1}{N}]\right) \\ &= \phi_t\ d\left(\frac{S_t}{N_t}\right) \\ &= \phi_t d\hat{S}_t \end{aligned}$$

□

**Corollary 1.6** *If a contingent claim is attainable in a given numéraire, it is also attainable in any other numéraire, and the replicating portfolio is the same.*

□

In particular, if the numéraire is the bank account, then

$$\bar{V}_t(\phi) = \bar{V}_0(\phi) + \int_0^t \phi_u\ d\bar{S}_u$$

**Remarks 1.7** A self–financing portfolio $\phi = (\phi^0, \ldots, \phi^N)$ is completely determined by the $N$ of the $N+1$ components. Thus if we are given the risky asset components $\phi^1, \ldots, \phi^N$,

the value of the riskless asset component $\phi^0$ is completely determined by the self–financing condition: Take $S^0$ to be the numéraire, so that

$$\phi_t \cdot \bar{S}_t = \bar{V}_t(\phi) = \bar{V}_0(\phi) + \sum_{n=0}^{N} \int_0^t \phi_u^n \, d\bar{S}_u^n = \bar{V}_0(\phi) + \sum_{n=1}^{N} \int_0^t \phi_u^n \, d\bar{S}_u^n$$

because $d\bar{S}_t^0 = 0$ — $\bar{S}_t^0 = 1$ is constant. Hence

$$\phi_t^0 = \bar{V}_0(\phi) + \sum_{n=1}^{N} \left[ \int_0^t \phi_u^n \, d\bar{S}_u^n - \phi_t^n \bar{S}_t^n \right]$$

□

## 1.2 Martingale Pricing

**Definition 1.8** *Suppose that $N$ is a numéraire. A measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ is an* equivalent martingale measure *(EMM) for numéraire $N$ if an only if*

*(i)* $\mathbb{Q} \sim \mathbb{P}$*;*

*(ii)* $\hat{S} = (\frac{S_t}{N_t})_t$ *is a (local) $\mathbb{Q}$–martingale.*

*If $\hat{S}_t$ is a $\mathbb{Q}$–martingale, $\mathbb{Q}$ is called a* strong *EMM.*
*An EMM associated with the money market account is called a* riskneutral measure*.*

□

If $N$ is a numéraire, define $\hat{V}_t(\phi) = \frac{V_t(\phi)}{N_t}$, and define

$$\hat{G}_t(\phi) = \int_0^t \phi_u \, d\hat{S}_u$$

Note that if $\mathbb{Q}$ is an EMM for $N$, then both $\hat{V}$ and $\hat{G}$ are $\mathbb{Q}$–local martingales. Indeed, $\hat{G}_t = \int_0^t \phi_u \, d\hat{S}_u$ is a sum of stochastic integrals w.r.t. a $\mathbb{Q}$–local martingale.

We require $\mathbb{Q}$ to be equivalent to $\mathbb{P}$ so that both measures have the same arbitrage strategies: $\mathbb{P}(G_T > 0) > 0$ if and only if $\mathbb{Q}(G_T > 0) > 0$.

Further note that

- An arbitrage opportunity remains an arbitrage under
  - a change of equivalent measure;
  - a change of numéraire.
- A replicating portfolio remains a replicating portfolio under
  - a change of equivalent measure;
  - a change of numéraire.

**Theorem 1.9** *If an EMM $\mathbb{Q}$ exists (for some numéraire $N$), then there are no arbitrage opportunities.*

**Proof:** If $\phi$ is a self–financing strategy, then

$$0 = \hat{G}_0(\phi) = \mathbb{E}_{\mathbb{Q}}[\hat{G}_T(\phi)]$$

Now because $\mathbb{P}(\hat{G}_T > 0) > 0$ if and only if $\mathbb{Q}(\hat{G}_T > 0) > 0$, and because $G_T \geq 0$ if and only if $\hat{G}_T \geq 0$, we cannot have both $G_T \geq 0$ and $\mathbb{E}_{\mathbb{P}}[G_T] > 0$. Thus $\phi$ cannot be an arbitrage, i.e. there are no arbitrage opportunities.

□

**Example 1.10** The most common choice of numéraire is the money market account. Suppose that $S_t^0$ is the MMA, with price dynamics

$$dS_t^0 = r(t,\omega)S_t^0 \, dt$$

If $\mathbb{Q}$ is the EMM associated with $S^0$, then each $\bar{S}_t^n$ is a $\mathbb{Q}$– local martingale. Now

$$d\bar{S}_t^n = \frac{dS_t^n}{S_t^0} - \frac{S_t^n}{(S_t^0)^2} dS_t^0$$

and so

$$dS_t^n = S_t^0 d\bar{S}_t^n + rS_t^n \, dt = rS_t^n \, dt + dM_t^n$$

where $M_t^n = \int_0^t S_t^0 \, d\bar{S}_t^n$ is a $\mathbb{Q}$–local martingale. Conversely, if each $dS_t^n = rS_t^n \, dt + dM_t^n$ for some $\mathbb{Q}$–local martingale $M_t^n$, then $\mathbb{Q}$ is a riskneutral measure.

□

**Theorem 1.11** (Riskneutral Valuation)
*Suppose that $X$ is an attainable contingent claim, and that $\mathbb{Q}$ is an EMM for numéraire $N$. Then*

$$\hat{X}_t = \mathbb{E}_{\mathbb{Q}}[\hat{X}_T|\mathcal{F}_t]$$

*i.e.*

$$X_t = N_t \mathbb{E}_{\mathbb{Q}}\left[\frac{X_T}{N_T}|\mathcal{F}_t\right]$$

**Proof:** If $\phi$ replicates $X$, it does so under any numéraire, any EMM. Now by the Law of One Price,

$$\hat{X}_t = \hat{V}_t(\phi) = \mathbb{E}_{\mathbb{Q}}[\hat{V}_T(\phi)] = \mathbb{E}_{\mathbb{Q}}[\hat{X}_T]$$

□

## 1.3 Introduction to Change of Numéraire

Thus far, we've mainly considered two probability measures, the "real world" measure $\mathbb{P}$, and the equivalent martingale measure $\mathbb{Q}$ for the money market account numéraire. We've seen, however, that it is possible to introduce an EMM for different numéraires, and to use these for pricing. We now show that a change of numéraire is a technique which often simplifies a pricing problem – it is analogous to a reduction in dimension.

Consider an interest rate derivative $X$, and let $A_t$ be the bank account. If $\mathbb{Q}$ is the EMM for $A_t$, then $\frac{X_t}{A_t}$ is a $\mathbb{Q}$–martingale (assuming, of course, that $X$ is attainable).Thus

$$X_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s,\omega)\,ds}\;X(T)\right]$$

where $r$ is the short rate, so that $A_t = e^{\int_0^t r(s,\omega)\,ds}$. In order to compute this, we would have to know the joint density of $X(T)$, $A(T)$ under $\mathbb{Q}$ — it would not be observable, because only $\mathbb{P}$–densities can be observed. The computation of the expectation would involve a double integral.

The reason we haven't noticed this problem before is that we generally assumed that interest rates are constant, which simplifies matters considerably. If we assume that the payoff $X(T)$ and the short rate are independent under $\mathbb{Q}$, then we would still have some simplification, namely

$$\begin{aligned} X_0 &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s,\omega)\,ds}\right]\mathbb{E}_{\mathbb{Q}}\left[X(T)\right] \\ &= p(0,T)\mathbb{E}_{\mathbb{Q}}\left[X(T)\right] \end{aligned}$$

where $p(t,T)$ is the time $t$–price of a zero coupon bond with face value 1 and maturity $T$, i.e. an interest rate derivative with payoff 1 at expiry, in all states of the world. The above expression is obviously much simpler:

- It only involves a single integral, and needs only the $\mathbb{Q}$–density of $X(T)$.
- $p(0,T)$ is observable (either directly, or by bootstrapping a yield curve from observable coupon bond prices).

Generally, of course, $X(T), A(T)$ are *not* independent under $\mathbb{Q}$. Even if they *were* independent under $\mathbb{P}$, they would nevertheless probably not be independent under $\mathbb{Q}$ — under $\mathbb{Q}$, the drifts of all assets are the same, namely the short rate. Thus $X_t$ has the same drift as $A_t$, implying some correlation.

## 1.4 Mechanics of Changes of Numéraire

As usual, we work with a market model $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t, (S_t^0, \ldots, S_t^N)_t)$. Recall:

- A *numéraire* is a traded asset (posibly a portfolio of assets) with a strictly positive price process.
- Self–financing portfolios remain self–financing under a change of numéraire.
- Replicating portfolios remain replicating portfolios.

This is also a good time to recall:

**Theorem 1.12** (Bayes' Theorem)
*Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a filtration $\mathcal{F}_n$, and that $\mathbb{Q} << \mathbb{P}$. Let $\xi = d\mathbb{Q}/d\mathbb{P}$ and likelihood process $\xi_t = \mathbb{E}_{\mathbb{P}}[\xi|\mathcal{F}_t]$. Then*

*(a) For any random variable $Z$ (integrable w.r.t. $\mathbb{P}$ and $\mathbb{Q}$) we have*

$$\xi_t \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}}[Z\xi|\mathcal{F}_t]$$

*(b) If $\mathbb{Q} \approx \mathbb{P}$, then a stochastic process $X_t$ is a martingale under $\mathbb{Q}$ if and only if $\xi_t X_t$ is a martingale under $\mathbb{P}$.*

The proof is an exercise:

**Exercises 1.13** Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a filtration $\mathcal{F}_n$, and that $\mathbb{Q} << \mathbb{P}$. Let $\xi = d\mathbb{Q}/d\mathbb{P}$ and define $\xi_t = \mathbb{E}_{\mathbb{P}}[\xi|\mathcal{F}_t]$, where $\mathbb{E}_{\mathbb{P}}$ refers to *expectation w.r.t. the measure* $\mathbb{P}$.

(a) Show that for any random variable $Z$ (integrable w.r.t. $\mathbb{P}$ and $\mathbb{Q}$) we have

$$\xi_t \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}}[Z\xi|\mathcal{F}_t]$$

(b) Show that if $\mathbb{Q} \approx \mathbb{P}$, then a stochastic process $X_t$ is a martingale under $\mathbb{Q}$ if and only if $\xi_t X_t$ is a martingale under $\mathbb{P}$.

[Hint: (a) I'll give you the proof. You justify every step: Let $A \in \mathcal{F}_n$. Then

$$\begin{aligned}
\int_A \xi_t \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_t]\, d\mathbb{P} &= \int_A \mathbb{E}_{\mathbb{P}}[\xi \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_t]|\mathcal{F}_t]\, d\mathbb{P} \\
&= \int_A \xi \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_t]\, d\mathbb{P} \\
&= \int_A Z\, d\mathbb{Q} \\
&= \int_A Z\xi\, d\mathbb{P}
\end{aligned}$$

(b) Use (a). ]

□

Now the bank account $S_t^0 = A_t$ is just a special numéraire — one whose dynamics have zero volatility: $dA_t = r(t,\omega)A_t\, dt$. Let $\mathbb{Q}$ be the EMM for $A_t$. Then each $\frac{S_t^n}{A_t}$ is a $\mathbb{Q}$–martingale, i.e. $\mathbb{Q}$ "martingalizes" the ratios $\frac{S_t^n}{A_t}$.

Suppose that $\hat{A}(t)$ is another numéraire, with EMM $\hat{\mathbb{Q}}$. $\hat{\mathbb{Q}}$ "martingalizes" the ratios $\frac{S_t^n}{\hat{A}_t}$. Given that $\hat{A}(t)$ is a (combination of) traded assets, we expect $\frac{\hat{A}(t)}{A(t)}$ to be a $\mathbb{Q}$–martingale as well. If $X$ is an attainable claim, then $\frac{X_t}{A_t}$ is a $\mathbb{Q}$–martingale, and $\frac{X_t}{\hat{A}_t}$ is a $\hat{\mathbb{Q}}$–martingale.

What does $\hat{\mathbb{Q}}$ look like? Since $\mathbb{Q}, \hat{\mathbb{Q}}$ are both equivalent to $\mathbb{P}$, they are equivalent to each other, and thus the Radon–Nikodym derivative

$$L(T) = \frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}$$

exists. We don't know yet what it is, though, because we don't know $\hat{\mathbb{Q}}$. Nevertheless, it exists, so we may define the likelihood process

$$L(t) = \mathbb{E}_{\mathbb{Q}}[L(T)|\mathcal{F}_t]$$

We have, by Bayes' Theorem,

$$\frac{X_0}{\hat{A}_0} = \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{X(T)}{\hat{A}(T)}\right] = L(0)^{-1}\mathbb{E}_{\mathbb{Q}}\left[\frac{X(T)}{A(T)}L(T)\right]$$

so that

$$\frac{X_0}{A_0} = \mathbb{E}_{\mathbb{Q}}\left[\frac{X(T)}{A(T)}\frac{\hat{A}_0}{A_0}L(T)\right]$$

because $L(0) = 1$. But

$$\frac{X_0}{A_0} = \mathbb{E}_{\mathbb{Q}}\left[\frac{X(T)}{A(T)}\right]$$

This suggests that we turn every thing around and *define*

$$L(T) = \frac{\hat{A}(T)/A(T)}{\hat{A}(0)/A(0)}$$

and then *define* $\hat{\mathbb{Q}}$ by

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} = L(T)$$

Then $L(t) = \frac{\hat{A}_t/A_t}{\hat{A}_0/A_0}$, as you can easily check.

In general, we may use for $\hat{A}(t)$ absolutely any process with the property that $\frac{\hat{A}_t}{A_t}$ is a strictly positive $\mathbb{Q}$–martingale.

**Theorem 1.14** (Martingale Measure Pricing)
*Suppose that $\hat{A}(t)$ is process with the property that $\frac{\hat{A}_t}{A_t}$ is a strictly positive $\mathbb{Q}$–martingale. Define*

$$L(t) = \frac{\hat{A}_t/A_t}{\hat{A}_0/A_0} \qquad \frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}} = L(T)$$

*If $\frac{M_t}{A_t}$ is a $\mathbb{Q}$–martingale, then $\frac{M_t}{\hat{A}_t}$ is a $\hat{\mathbb{Q}}$–martingale. In particular, if $X$ is an attainable contingent claim, then*

$$X_t = \hat{A}_t \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{X(T)}{\hat{A}(T)}|\mathcal{F}_t\right]$$

□

In fact, we can generalize even more:

**Theorem 1.15** *Suppose that $\alpha_1(t), \alpha_2(t)$ are numeraires, and that $\mathbb{Q}_1, \mathbb{Q}_2$ are their associated EMM's. Then for any random variable $X$ we have*

$$\alpha_1(t)\mathbb{E}_{\mathbb{Q}_1}\left[\frac{X}{\alpha_1(T)}|\mathcal{F}_t\right] = \alpha_2(t)\mathbb{E}_{\mathbb{Q}_2}\left[\frac{X}{\alpha_2(T)}|\mathcal{F}_t\right]$$

**Proof:** Define the likelihood process $L_1(t) = \frac{\alpha_1(t)/A(t)}{\alpha_1(0)/A(0)}$, and define $L_2(t)$ similarly. Then by Bayes' Theorem

$$\begin{aligned}\alpha_1(t)\mathbb{E}_{\mathbb{Q}_1}\left[\frac{X}{\alpha_1(T)}|\mathcal{F}_t\right] &= \alpha_1(t)L_1(t)^{-1}\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{\alpha_1(T)}L_1(T)|\mathcal{F}_t\right] \\ &= A(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{A(T)}|\mathcal{F}_t\right]\end{aligned}$$

and the same goes for $\alpha_2$.

□

Let's investigate how asset price dynamics change when we move from $\mathbb{Q}$–world to $\hat{\mathbb{Q}}$–world. Assuming that asset prices are Ito diffusions, we have $\mathbb{Q}$–dynamics

$$dS_t = D[S_t]r_t\,dt + D[S_t]\sigma_t\,dW_t$$

where $W$ is a ($K$–dimensional) $\mathbb{Q}$–Brownian motion. The Radon–Nikodym derivative (process) which effects the change from $\mathbb{Q}$ to $\hat{\mathbb{Q}}$ is

$$L(t) = \frac{\hat{A}(t)/A(t)}{\hat{A}(0)/A(0)}$$

Using the fact that $\hat{A}_t/A_t$ is a $\mathbb{Q}$–martingale, we see that

$$d\hat{A}_t = r_t\hat{A}_t\,dt + \hat{\sigma}_t\hat{A}_t\,dW_t$$

By Ito's formula,

$$dL_t = \frac{A_0}{\hat{A}_0}\left[\frac{1}{A_t}\left(r_t\hat{A}_t\,dt + \hat{\sigma}_t\hat{A}_t\,dW_t\right) - \frac{\hat{A}_t}{A_t^2}\left[r_tA_t\,dt\right)\right]$$

Thus

$$dL_t = L_t\hat{\sigma}_t\,dW_t$$

confirming that $L(t)$ is a $\mathbb{Q}$–martingale, as we already knew. Solving the SDE, we obtain

$$L(t) = e^{\int_0^t \hat{\sigma}_s\,dW_s - \frac{1}{2}\int_0^t ||\hat{\sigma}_s||^2\,ds}$$

and thus

> Suppose we change the numéraire from the MMA $A_t$ to $\hat{A}_t$. Then the EMM $\hat{\mathbb{Q}}$ associated with $\hat{A}_t$ is obtained from the EMM $\mathbb{Q}$ associated with $A_t$ by a Girsanov transformation whose kernel is the volatility $\hat{\sigma}$ of the new numéraire $\hat{A}_t$.

By Girsanov's Theorem

$$\hat{W}_t = W_t - \int_0^t \hat{\sigma}_s\,ds$$

is a ($K$–dimensional) $\hat{\mathbb{Q}}$–Brownian motion, and thus the asset dynamics under $\hat{\mathbb{Q}}$ are given by

$$dS_t = D[S_t] \begin{pmatrix} r_t + \sigma_t^1 \cdot \hat{\sigma}_t \\ \vdots \\ r_t + \sigma_t^N \cdot \hat{\sigma}_t \end{pmatrix} dt + D[S_t]\sigma \, d\hat{W}_t$$

i.e.

$$dS_t^n = (r_t + \sigma_t^n \cdot \hat{\sigma})S_t^n \, dt + \sigma_t^n S_t^n \, d\hat{W}_t$$

where $\sigma^n$ is the $n^{\text{th}}$ row of the volatility matrix $\sigma$. In particular, the "discounted" asset ratios $\hat{S}_t^n = \frac{S_t^n}{\hat{A}_t}$ have dynamics

$$d\hat{S}_t^n = \hat{S}_t^n(\sigma_t^n - \hat{\sigma}_t) \, d\hat{W}^t$$

as you can easily verify by applying Ito's formula. Hence the $\hat{S}_t^n$ are $\hat{\mathbb{Q}}$–martingales, and the volatility of each "discounted" asset is reduced by the volatility of the numéraire.

**Remarks 1.16** Consider a simple Black–Scholes model, where the risky asset prices are given by a geometric Brownian motions, driven by a single source of noise. The market price of risk of $S^n$ in $\hat{\mathbb{Q}}$–world is

$$\frac{r + \sigma^n \hat{\sigma} - r}{\sigma^n} = \hat{\sigma}$$

i.e. all assets have the same market price of risk, namely the volatility of the numéraire. This is also true in the multidimensional case, where the market price of risk is a vector.

The bank account has zero volatility, and thus the MPR in $\mathbb{Q}$–world is zero.

□

## 1.5 A General Option pricing Formula

Consider a call $C$ on a security $S$ with strike $K$ and maturity $T$. Let $\mathbb{Q}_S$ be the EMM associated with numéraire $S$, and let $\mathbb{Q}^T$ be the $T$–forward measure (i.e. the EMM associated with the zero coupon bond $p(t,T)$ maturing at $T$).

**Theorem 1.17**

$$C_0 = S_0 \mathbb{Q}_S(S_T \geq K) - KP(0,T)\mathbb{Q}^T(S_T \geq K)$$

**Proof:** We have

$$\begin{aligned} C_0 &= p(0,T)\mathbb{E}_{\mathbb{Q}^T}[\max\{S_T - K, 0\}] \\ &= p(0,T)\mathbb{E}_{\mathbb{Q}^T}[S_T - K; S_T \geq K] \\ &= p(0,T)\mathbb{E}_{\mathbb{Q}^T}[S_T; S_T \geq K] - Kp(0,T)\mathbb{Q}^T(S_T \geq K) \end{aligned}$$

But we have

$$\alpha_1(t)\mathbb{E}_{\mathbb{Q}_1}\left[\frac{X}{\alpha_1(T)}|\mathcal{F}_t\right] = \alpha_2(t)\mathbb{E}_{\mathbb{Q}_2}\left[\frac{X}{\alpha_2(T)}|\mathcal{F}_t\right]$$

for general numéraires and their associated EMM's. Using this with $\alpha_1(t) = p(t,T)$ and $\alpha_2(t) = S_t$, we obtain

$$\begin{aligned} p(0,T)\mathbb{E}_{\mathbb{Q}^T}[S_T; S_T \geq K] &= p(0,T)\mathbb{E}_{\mathbb{Q}^T}\left[\frac{S_T I_{\{S_T \geq K\}}}{p(T,T)}\right] \\ &= S_0 \mathbb{E}_{\mathbb{Q}_S}\left[\frac{S_T I_{\{S_T \geq K\}}}{S_T}\right] \\ &= S_0 \mathbb{Q}_S(S_T \geq K) \end{aligned}$$

□

## 1.6 Applications

**Example 1.18** Forward Measures
Consider again the situation at the beginning of this chapter: We consider a contingent claim $X$ with expiry $T$. Under riskneutral valuation, its value is

$$X_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s,\omega)\,ds}\,X(T)\right]$$

where $r$ is the short rate. We bemoaned the fact that this would necesssitate us knowing the joint density of $A_T$ and $X_T$. If only, we said, $A_T$ and $X_T$ were independent, we would get the much simpler

$$\begin{aligned} X_0 &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s,\omega)\,ds}\right]\mathbb{E}_{\mathbb{Q}}\left[X(T)\right] \\ &= p(0,T)\mathbb{E}_{\mathbb{Q}}\left[X(T)\right] \end{aligned}$$

where $p(t,T)$ is the time $t$–price of a zero coupon bond with face value 1 and maturity $T$. If only. . .

Now let's see what happens if we change the numéraire to $p(t,T)$, and let $\mathbb{Q}_T$ be the corresponding EMM. In that case, the pricing formula becomes

$$\frac{X_0}{p(0,T)} = \mathbb{E}_{\mathbb{Q}_T}\left[\frac{X_T}{p(T,T)}\right]$$

and noting that $P(T,T)=1$, we have

$$X_0 = p(0,T)\mathbb{E}_{\mathbb{Q}_T}\left[X_T\right]$$

This is the simple form that we sought, but it's correct under $\mathbb{Q}_T$, and *not* under $\mathbb{Q}$.

The measure $\mathbb{Q}_T$ is called the *T–forward measure.* Note that if interest rates are deterministic, then $\mathbb{Q}$ and $\mathbb{Q}_T$ coincide, because then $p(t,T) = e^{-\int_t^T r_s\,ds}$, so that each ratio $S_t^n/p(t,T) = A_T(S_t^n/A_t) = \text{const.} \times S_t^n/A_t$ is already a $\mathbb{Q}$–martingale.

However, when interest rates are stochastic, $\mathbb{Q}$ and $\mathbb{Q}_T$ are quite different. We shall see later that futures prices are $\mathbb{Q}$–martingales, whereas forward prices are $\mathbb{Q}_T$–martingales. Thus forward prices and futures prices coincide if interest rates are deterministic.

□

**Example 1.19** Exchange Options
Consider an *exchange option* which gives the right, but not the obligation, to exchange asset $S^1$ for asset $S^2$ at time $T$. This is a contingent claim $X$ with payoff

$$X_T = \max\{S_T^2 - S_T^1, 0\}$$

Using riskneutral valuation (i.e. MMA as numéraire), its value is therefore

$$X_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_t\,dt}\max\{S_T^2 - S_T^1, 0\}\right]$$

To compute this, we have to know the joint distributions of $A_T, S_T^1, S_T^2$ under $\mathbb{Q}$, yielding a triple integral.

It is computationally simpler to change the numéraire: Let $\hat{A}_t = S^1_T$, and let $\hat{\mathbb{Q}}$ be the associated EMM. The contingent cliam is then priced as follows:

$$\frac{X_0}{S^1_0} = \mathbb{E}_{\hat{\mathbb{Q}}}\left[\frac{\max\{S^2_T - S^1_T, 0\}}{S^1_T}\right] = \mathbb{E}_{\hat{\mathbb{Q}}}\left[\max\{\hat{S}^2_T - 1, 0\}\right]$$

(where $\hat{S}^2_t = \frac{S^2_t}{S^1_t}$). This looks like a call on $\hat{S}^2$ with strike $K = 1$, and we only have to know the distribution of $\hat{S}^2_T$ under $\hat{\mathbb{Q}}$.

To price this option, we have to assume some form of asset dynamics. Suppose these are given by one–dimensional Ito diffusions, i.e. suppose we have $\mathbb{P}$–dynamics

$$\begin{aligned} dS^1_t &= \mu_1 S^1_t\, dt + \bar{\sigma}_1\, d\bar{W}^1_{\mathbb{P}}(t) \\ dS^2_t &= \mu_2 S^2_t\, dt + \bar{\sigma}_2\, d\bar{W}^2_{\mathbb{P}}(t) \end{aligned}$$

where $\bar{W}^1_{\mathbb{P}}, \bar{W}^2_{\mathbb{P}}$ are *correlated* $\mathbb{P}$–Brownian motions, with correlation $\rho_t$. To get lognormality we also assume that $\bar{\sigma}_1(t), \bar{\sigma}_2(t)$ and $\rho(t)$ are *deterministic*. Interest rates, however, can be stochastic.

We can write this as

$$d\begin{pmatrix} S^1_t \\ S^2_t \end{pmatrix} = \begin{pmatrix} \mu_1 S^1_t \\ \mu_2 S^2_t \end{pmatrix} dt + D[S_t]\sigma_t\, dW_{\mathbb{P}}(t)$$

where $W^1_{\mathbb{P}}, W^2_{\mathbb{P}}$ are independent $\mathbb{P}$–Brownian motions and

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Then we must have

$$\begin{aligned} \bar{\sigma}_1 &= \sqrt{\sigma^2_{11} + \sigma^2_{12}} \\ \bar{\sigma}_2 &= \sqrt{\sigma^2_{21} + \sigma^2_{22}} \quad \text{and} \\ \rho &= \frac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sqrt{\sigma^2_{11} + \sigma^2_{12}}\sqrt{\sigma^2_{21} + \sigma^2_{22}}} \end{aligned}$$

So given $\bar{\sigma}_1, \bar{\sigma}_2$ and$\rho$ we can solve for the matrix $\sigma$ (though not uniquely).

Note that the correlation s a function of the volatility matrix. When we change from $\mathbb{P}$–world to $\mathbb{Q}$–world, the volatility matrix is unchanged, and thus also the correlation. Thus under $\mathbb{Q}$, the asset dynamics are

$$\begin{aligned} dS^1_t &= r_t S^1_t\, dt + \bar{\sigma}_1\, d\bar{W}^1_{\mathbb{Q}}(t) \\ dS^2_t &= r_t S^2_t\, dt + \bar{\sigma}_2\, d\bar{W}^2_{\mathbb{Q}}(t) \end{aligned}$$

where $\bar{W}^1_{\mathbb{Q}}, \bar{W}^2_{\mathbb{Q}}$ are $\mathbb{Q}$–Brownian motions, with correlation $\rho$. This can also be written as

$$d\begin{pmatrix} S^1_t \\ S^2_t \end{pmatrix} = D[S_t]r_t\, dt + D[S_t]\sigma_t\, dW_{\mathbb{Q}}(t)$$

where $W^1_{\mathbb{Q}}, W^2_{\mathbb{Q}}$ are independent $\mathbb{Q}$–Brownian motions. Now when we change the numéraire from the MMA to $S^1$, and the measure from $\mathbb{Q}$ to $\hat{\mathbb{Q}}$, we get

$$\begin{aligned} d\hat{S}^2_t &= \hat{S}^2_t(\sigma_{21}-\sigma_{11}, \sigma_{22}-\sigma_{12}) \cdot \ d\begin{pmatrix} W^1_{\hat{\mathbb{Q}}}(t) \\ W^2_{\hat{\mathbb{Q}}}(t) \end{pmatrix} \\ &= \hat{S}^2_t \hat{\sigma}_2 \, d\bar{W}_{\hat{\mathbb{Q}}}(t) \end{aligned}$$

where $\bar{W}_{\hat{\mathbb{Q}}}$ is a one–dimensional $\hat{\mathbb{Q}}$–Brownian motion and

$$\hat{\sigma}_2 = \sqrt{(\sigma_{21}-\sigma_{11})^2 + (\sigma_{22}-\sigma_{12})^2} = \sqrt{(\bar{\sigma}_1)^2 + (\bar{\sigma}_2)^2 - 2\rho\bar{\sigma}_1\sigma_2}$$

Now since the $\bar{\sigma}_1(t), \bar{\sigma}_2(t)$ and $\rho(t)$ are assumed to be deterministic, so is $\hat{\sigma}_2(t)$. It follows that $\hat{S}^2_T$ is lognormal under $\hat{\mathbb{Q}}$: Indeed

$$\hat{S}^2_T = \hat{S}^2_0 e^{\int_0^T \hat{\sigma}_2(t)\, d\bar{W}_{\hat{\mathbb{Q}}}(t) - \frac{1}{2}\int_0^T \hat{\sigma}_2(t)^2\, dt}$$

so that

$$\ln(S^2_T/S^2_0) \sim N\left(-\frac{1}{2}\int_0^T \hat{\sigma}_2(t)^2\, dt, \frac{1}{2}\int_0^T \hat{\sigma}_2(t)^2\, dt\right)$$

Using the properties of lognormality, we see that

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left[\max\{\hat{S}^2_T - 1, 0\}\right] = \mathbb{E}_{\hat{\mathbb{Q}}}[\hat{S}^2_T]N(d_1) - 1N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(\frac{\mathbb{E}_{\hat{\mathbb{Q}}}[\hat{S}^2_T]}{1}) + \frac{1}{2}\int_0^T \hat{\sigma}_2(t)^2\, dt}{\sqrt{\int_0^T \hat{\sigma}_2(t)^2\, dt}} \\ d_2 &= d_1 - \sqrt{\int_0^T \hat{\sigma}_2(t)^2\, dt} \end{aligned}$$

and thus, using the fact that $\mathbb{E}_{\hat{\mathbb{Q}}}[\hat{S}^2_T] = \hat{S}^2_0$, we have

$$X_0 = S^2_0 N(d_1) - S^1_0 N(d_2)$$

If we further assume that $\bar{\sigma}_1, \bar{\sigma}_2$ and $\rho$ are constants, we obtain

$$\begin{aligned} X_0 &= S^2_0 N(d_1) - S^1_0 N(d_2) \\ \text{where} \quad d_1 &= \frac{\ln(\frac{S^2_0}{S^1_0}) + \frac{1}{2}(\bar{\sigma}_1^2 + \bar{\sigma}_2^2 - 2\rho\bar{\sigma}_1\bar{\sigma}_2)T}{\sqrt{\bar{\sigma}_1^2 + \bar{\sigma}_2^2 - 2\rho\bar{\sigma}_1\bar{\sigma}_2)T}} \\ d_2 &= d_1 - \sqrt{\bar{\sigma}_1^2 + \bar{\sigma}_2^2 - 2\rho\bar{\sigma}_1\bar{\sigma}_2)T} \end{aligned}$$

where we used the fact that

$$\hat{\sigma}_2 = \sqrt{(\bar{\sigma}_1)^2 + (\bar{\sigma}_2)^2 - 2\rho\bar{\sigma}_1\sigma_2}$$

$\square$

Note that, in the above example, $\frac{1}{T}\int_0^T \hat{\sigma}_2(t)^2\, dt$ is just the average of the squared volatility of $\hat{S}^2$, so that $\int_0^T \hat{\sigma}_2(t)^2\, dt = \sigma^2_{\text{average}} T$.

**Example 1.20** A convertible bond is a bond (issued by a corporation) that can be converted to equity at certain times, using a predetermined exchange ratio. For example, a bond with a conversion ratio of 10 allows its holder to convert a par \$1000 bond to 10 shares of the common stock, at some future date $T$ (which may, or may not, e the maturity of the bond). Thus

$$\text{Convertible bond} = \text{regular bond} + 10 \text{ calls with strike } \frac{\text{Bond price}}{10}$$

Here the bond price is the future bond price at the convertion date $T$, which is unknown if $T$ is not also the maturity of the bond. The strike varies because the bond price varies.

Thus a convertible bond will be difficult to price if interest rates are stochastic, and practitioners will generally use an estimate of the future bond price. For example, if the bond's coupon rate is approximately equal to the prevailing interest rate, then setting the future bond price equal to its par value will not be a bad approximation. The fact that the corporation typically issues new shares to honour the conversion also means that a dilution effect must be taken into account.

We make the following simplifying assumptions:

- The bond is a zero coupon bond, with face = 1, maturity $T_1$;
- The underlying stock pays no dividends;
- At a fixed date $T_0 < T_1$, the bond can be converted to $c$b shares of stock.

Our aim is to price this convertible bond at some time $t < T_0$.

The payoff of this bond is

$$X_{T_0} = \max\{cS_{T_0}, p(T_0, T_1)\}$$

where $p(t, T_1)$ is a face = 1 zero coupon bond of the same risk class as the convertible bond.

In order to simplify this expression, we should use either $S$ or $p(t, T_1)$ as numeraire. In the first case, we obtain

$$X_t = S_t \mathbb{E}_{\mathbb{Q}_S}\left[\max\{c, \frac{p(T_0, T_1)}{S_{T_0}}\}|\mathcal{F}_t\right]$$

whereas the second choice yields

$$X_t = p(t, T_1)\mathbb{E}_{\mathbb{Q}^{T_1}}\left[\max\{\frac{cS_{T_0}}{p(T_0, T_1)}, 1\}|\mathcal{F}_t\right]$$

Let's consider the second possibility: Define $\hat{S}_t = \frac{S_t}{p(t,T_1)}$. Then

$$X_t = cp(t, T_1)\mathbb{E}_{\mathbb{Q}^{T_1}}\left[\max\{\hat{S}_{T_0} - 1/c, 0\}|\mathcal{F}_t\right] + p(t, T_1)$$

Now we can observe the market value of $p(t, T_1)$, so we need only calculate

$$cp(t, T_1)\mathbb{E}_{\mathbb{Q}^{T_1}}\left[\max\{\hat{S}_{T_0} - 1/c, 0\}|\mathcal{F}_t\right]$$

which is just the price of $c$ calls with strike $\frac{1}{c}$.

Now in order to price the option, we need to know something about the distribution of $\hat{S}_{T_0}$. This requires the specification of both a stock price model and an interest rate model. For the stock price model, we can take the usual GBM. As for interest rate models, this is the subject we will tackle next. We will continue this example in the Exercises, where you will be asked to price a convertible zero–coupon bond in the Ho–Lee model.

□

# 2 Modelling Fixed Income: Introduction

## 2.1 Classification of Interest Rate Models

We will examine several approaches for the modelling of interest rates: short rate modelling, whole yield curve modelling and market models. The purpose of this section is to introduce basic concepts and notation. Amongst the immediately obvious quantities that we may model are

- bond prices
- the short rate
- forward rates (discretely or continuously compounded)
- the entire yield curve

Of course, a model of bond prices will have the yield curve as an output, etc. These approaches are no independent.

**Short rate models:** These model just one variable, the short rate, which is an idealized quantity that represents the instantaneous interest rate at any time. Usually a diffusion model, and thus Markov. We specify dynamics, e.g.

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t \qquad \kappa, \theta, \sigma \text{ const.}$$

is the Vasiček model, and

$$dr_t = \kappa(\theta(t) - r_t)\,dt + \sigma(t)\,dW_t \qquad \kappa \text{ const.}, \theta, \sigma \text{ deterministic}$$

is the Hull–White (extended Vasiček) model.

- Can be one–factor or multi–factor
- *Affine* term structure models have a particularly simple form, allowing for closed form solutions for bond option prices, Eurodollar futures, etc. More later…
- Multi–factor models: principal component analysis shows that 80-90% of the variance of the term structure is explained by parallel shifts of the yield curve, 5-10% by a twist (long term and short term rates move in opposite directions, pivoting about a point), and 1-2% by a butterfly (long and short term rates move in the same direction, with mid–term rates moving in the opposite direction).

**Whole yield curve models:** These model the entire term structure of rates, eg. the entire forward rate curve. Examples are

- Heath–Jarrow–Morton models
- Market models

Interest rate models are often categorized into *Equilibrium models* and *No–arbitrage models*. Equilibrium models attempt to derive, e.g., short rate dynamics from macroeconomic considerations, starting from a representative investor (e.g. Cox–Ingersoll–Ross, Vasiček, Merton models). These models often have the nice property of being time–homogeneous, but usually are unable to fit observed prices exactly. No–arbitrage models attempt to fit a model exactly to observed prices and volatilities –zero coupon bonds, caplets, swaptions. (e.g. Ho–Lee, Hull–White models).

Both terms are misnomers: Some equilibrium models are not arbitrage–free, and thus *not* in equilibrium. Some no–arbitrage models permit negative interest rates, thus allowing "mattress arbitrage" (borrow from the bank when rates go negative, put under mattress).

## 2.2 Bond Market Basics

One of the basic instruments that we shall be concerned with is the following:

**Definition 2.1** A $T$–bond is a zero coupon bond with face value 1.00 and maturity $T$. Its value at time $t \leq T$ is denoted by $p(t,T)$.

□

These are also called discount bonds.

We work in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration which satisfies the usual conditions. We usually require that:

- $p(t,T)$ is a continuous semimartingale for each $T$.
- $0 \leq p(t,T) \leq 1$ a.s. (This fails in, e.g. Gaussian short rate models)
- There is a frictionless market for $T$–bonds of every maturity $T > 0$.
- For every fixed $T > 0$, $\{p(t,T) : 0 \leq t \leq T\}$ is an optional process with $P(T,T) = 1$.
- For every fixed $t$, $p(t,T)$ is ($\mathbb{P}$–a.s.) differentiable in the second variable $T$.

$$p_T(t,T) = \frac{\partial p(t,T)}{\partial T}$$

- No default risk.

Note that, for fixed $t$, the set $\{p(t,T) : T \geq 0\}$ is just the term structure of zero coupon bond prices, which is typically a smooth decreasing function (of $T$). On the other hand, for fixed $T$, the set $\{p(t,T) : t \leq T\}$ is the price process of the security $p(t,T)$, which is typically very ragged (i.e. of unbounded variation).

Note that there are, in our model, infinitely many securities, namely one $p(t,T)$ for each maturity $T$.

We briefly recall the definitions of the various types of rate:

- Let $t < S < T$. Consider the following strategy:
  - (i) At time $t$, short an $S$–bond, and use the proceeds to buy $\frac{p(t,S)}{p(t,T)}$–many $T$–bonds. Net cashflow at time $t$ is zero.
  - (ii) At time $S$, pay \$1.00 to redeem the $S$–bond.
  - (iii) At time $T$, receive $\frac{p(t,S)}{p(t,T)}$ from maturing $T$–bonds.

  Thus at time $T$, we can, with no initial cash outlay, ensure that a deposit of \$1.00 at time $S$ leads to a payoff of $\frac{p(t,S)}{p(t,T)}$ at time $T$. This implies that we can lock in an interest rate $R(t; S, T)$ for the future period $[S, T]$:

$$\begin{aligned} e^{R(T-S)} &= \frac{p(t,S)}{p(t,T)} \\ \Rightarrow R(t;S,T) &= -\frac{\ln p(t,T) - \ln p(t,S)}{T-S} \end{aligned}$$

  This is the *forward rate* (continuously compounded) for the period $[S, T]$ at time $t$.

- The equivalent *simple forward rate* (the LIBOR forward rate) for $[S, T]$ contracted at time $t$ is similarly defined by

$$\begin{aligned} 1 + L(T-S) &= \frac{p(t,S)}{p(t,T)} \\ \Rightarrow L(t;S,T) &= -\frac{p(t,T) - p(t,S)}{p(t,T)(T-S)} \end{aligned}$$

- The continuously and simple *spot rates* at time $t$ for time $T$ are $R(t; t, T)$ and $L(t; t, T)$ respectively.

- The *instantaneous forward rate* at time $t$ for time $T$ is the interest rate that can be locked in for an infinitesimal interval $[t, T + dT]$. It is given by

$$\begin{aligned} f(t,T) &= \lim_{\Delta t \to 0} R(t; T, T + \Delta T) \\ &= -\lim_{\Delta t \to 0} \frac{\ln p(t, T + \Delta T) - \ln p(t,T)}{\Delta T} \\ &= -\frac{\partial \ln p(t,T)}{\partial T} \end{aligned}$$

- The *short rate* is the instantaneous spot rate, and is defined by

$$r(t) = f(t,t)$$

- Given a *tenor structure*

$$0 \le t \le T_0 < T_1 < T_2 < \cdots < T_N$$

  we can find a *forward swap rate* $S_t = S(t; T_0, T_1, \ldots, T_N)$, the unique fixed rate, at time $t$, for which a fixed–for–floating forward swap, starting at time $T_0$, will have zero value. We clearly require, with $\tau_j = T_j - T_{j-1}$, that

$$\sum_{j=1}^{N} S_t \tau_j p(t, T_j) = \sum_{j=1}^{N} L(t; T_{j-1}, T_j) \tau_j p(t, T_j)$$

and thus

$$S_t = \frac{\sum_{j=1}^N L(t;T_{j-1},T_j)\tau_j p(t,T_j)}{\sum_{j=1}^N \tau_j p(t,T_j)}$$

But

$$\sum_{j=1}^N L(t;T_{j-1},T_j)\tau_j p(t,T_j) = \sum_{j=1}^N -[p(t,T_j) - p(t,T_{j-1})] = p(t,T_0) - p(t,T_N)$$

and hence

$$S_t = \frac{p(t,T_0) - p(t,T_N)}{\sum_{j=1}^N \tau_j p(t,T_j)}$$

The denominator $\sum_{j=1}^N \tau_j p(t,T_j)$ is sometimes referred to as the *value of a basis point.*

**Remarks 2.2** 1. The assumption that there are traded zero coupon bonds of every maturity is clearly false. Nevertheless, a large number of *implied* zero coupon bond prices can usually be obtained by bootstrapping the yield curve.

2. The instantaneous rates (forward– and short–) are *theoretical* entities, and not directly observable in the market. One of the shortcomings of short rate and HJM models is that they model these non-existent entities. *Market models* such as the BGM– and Jamshidian models, however, are concerned with the modelling of quoted market rates.

□

The following lemma shows how bond prices are related to forward rates:

**Lemma 2.3**

$$p(t,T) = p(t,S)e^{-\int_S^T f(t,u)\,du}$$

**Proof:** $\ln p(t,T) = \ln p(t,S) + \int_S^T \frac{\partial \ln p(t,u)}{\partial T}\,du.$

□

As usual, we denote that money market account (MMA) process $A_t$ by

$$A_t = e^{\int_0^t r(u)\,du}$$

where $r(t)$ is the short rate.

**Example 2.4** *No model that allows only parallel shifts of the yield curve is arbitrage–free.*
**Proof:** Suppose it is certain that $f(1,T) = f(0,T) + \varepsilon$ for all $T \geq 1$, where $\varepsilon$ is a random variable. Now choose times $1 < T_1 < T_2 < T_3$. At $t = 1$,

$$p(1,T) = e^{-\int_1^T f(1,u)\,du} = e^{-\int_1^T f(0,u)+\varepsilon\,du} = \frac{p(0,T)}{p(0,1)}e^{-\varepsilon(T-1)}$$

Now suppose that we hold $x_i$ $T_i$–bonds ($i = 1,2,3$). We construct an arbitrage, a static portfolio satisfying

(i) $\sum_{i=1}^3 x_i p(0,T_i) = 0$

(ii) $\sum_{i=1}^3 x_i p(1,T_i) > 0$ a.s.

At time 1 the value of the portfolio is

$$\begin{aligned} V_1(\varepsilon) &= \sum_{i=1}^3 x_i p(1,T_i) \\ &= \sum_{i=1}^3 x_i \frac{p(0,T_i)}{p(0,1)} e^{-\varepsilon(T_i-1)} \\ &= \sum_{i=1}^3 x_i \frac{p(0,T_i)}{p(0,1)} e^{-\varepsilon(T_i-T_2)} e^{-\varepsilon(T_2-1)} \\ &= g(\varepsilon)\frac{e^{-\varepsilon(T_2-1)}}{p(0,1)} \end{aligned}$$

where

$$g(\varepsilon) = \sum_{i=1}^3 x_i p(0,T_i) e^{-\varepsilon(T_i-T_2)}$$

We shall ensure that $V_1(\varepsilon) > 0$ whenever $\varepsilon \neq 0$. First note that $g(0) = 0$, because $\sum_{i=1}^3 x_i p(0,T_i) = 0$. Further, $V_1(\varepsilon)$ and $g(\varepsilon)$ always have the same sign, so to ensure $V_1(\varepsilon) > 0$, it suffices to ensure that $g(\varepsilon) > 0$.

Now $g$ is a $C^2$–function (twice differentiable), and we require that (i) $g(0) = 0$, (ii) $g(\varepsilon) > 0$ whenever $\varepsilon \neq 0$. It follows that $g'(0) = 0$, thus that

$$g'(0) = \sum_{i=1}^3 x_i (T_2 - T_i) p(0,T_i) = 0$$

and thus that

$$\sum_{i=1}^3 x_i T_i p(0,T_i) = 0$$

Next, if we ensure $g''(\varepsilon) > 0$, then, combined with $g(0) = g'(0) = 0$, we se that $g(\varepsilon) > 0$ for all $\varepsilon \neq 0$. Now

$$g''(\varepsilon) = \sum_{i=1}^3 x_i (T_2 - T_i)^2 p(0,T_i) e^{-\varepsilon(T_i-T_2)}$$

and thus $g''(\varepsilon) > 0$ for all $\varepsilon$ if $x_1, x_3 \geq 0$ (and at least one is $> 0$).

Now take $x_2 < 0$. Since $\sum_{i=1}^3 x_i p(0,T_i) = 0$, we see that at least one of $x_1, x_3$ must be $> 0$. Since $\sum_{i=1}^3 x_i (T_2 - T_i) p(0,T_i) p(0,T_i) = 0$, we see that $x_1, x_3$ have the same sign, i.e. both are $> 0$. Then $g''(\varepsilon) > 0$ for all $\varepsilon \neq 0$, and hence also $g(\varepsilon) > 0$.

It follows that any portfolio $(x_1, x_2, x_3)$ satisfying

(i) $\sum_{i=1}^3 x_i p(0,T_i) = 0$

(ii) $\sum_{i=1}^3 x_i (T_2 - T_i) p(0,T_i) = 0$

(iii) $x_2 < 0$

is an arbitrage.

□

**Example 2.5** Define the *long rate* $l(t)$ by

$$l(t) = \lim_{T\to\infty} R(t,T)$$

where $R(t,T)$ is the c.c. spot rate, i.e. $p(t,T) = e^{-R(t,T)}(T-t)$. Though $l(t)$ is not directly obtainable form traded securities (because the longest–term securities typically have a life of 30 years or so), it can be estimated, and empirical studies suggest that it fluctuates considerably over time. Most no–arbitrage models have a constant value for $l(t)$, however, and indeed

**Theorem:** *If the term–structure dynamics are arbitrage–free, then $l(t)$ is an increasing function a.s.*

**Proof:** By rescaling time, we may assume that $l(1) < l(0)$ with positive probability, to obtain a contradiction. For $T = 1, 2, 3, \ldots$, construct a portfolio which, at $t = 0$ invests $\frac{1}{T} - \frac{1}{T+1} = \frac{1}{T(T+1)}$ into each of the bonds $p(t,T)$, so that the value of then portfolio is $V_0 = \sum_{T=1}^{\infty} \frac{1}{T(T+1)} = 1$. Define $\varepsilon = (l(0) - l(1))/3$. Now $p(0,T) = e^{-r(0,T)T}$, and $r(0,T) \to l(0)$ as $T \to \infty$, so eventually, we have $r(0,T) > l(0,T) - \varepsilon$, i.e. $p(0,T) < e^{-(l(0)-\varepsilon)T}$ eventually. Similarly, $p(1,T) > e^{-(l(1)+\varepsilon)T}$ eventually. Suppose these relations hold for all $T \geq T_0$. Then

$$V_1 = \sum_{T=1}^{\infty} \frac{p(1,T)}{T(T+1)p(0,T)} > \sum_{T=1}^{T_0-1} \frac{p(1,T)}{T(T+1)p(0,T)} + \sum_{T=T_0}^{\infty} \frac{e^{\varepsilon T}}{T(T+1)}$$

The second term diverges to $\infty$, so that $V_1 = \infty$. Now since $V_0 = \mathbb{E}_{\mathbb{Q}}[V_1/B_1]$, where $\mathbb{Q}$ is a risk–neutral measure and $B$ is the bank account, we see that $\mathbb{Q}(V_1 = \infty) = 0$, because $V_0 = 1 < \infty$. Since the "real–world" measure $\mathbb{P}$ is equivalent to $\mathbb{Q}$, we must have $\mathbb{P}(V_1 = \infty) \leq \mathbb{P}(l(1) > l(0)) = 0$ as well.

□

## 2.3 Modelling the Bond Market

We consider three approaches:

1. Specify short rate dynamics;

2. Specify bond price dynamics;

3. Specify forward rate dynamics;

Suppose, for example, that we are given the following dynamics:

1. *Short rate dynamics:*

$$dr(t,\omega) = a(t,\omega)\, dt + b(t,\omega)\, dW_t$$

2. *Bond price dynamics:*

$$dp(t,T)(\omega) = p(t,T)(\omega)[m(t,T,\omega)\ dt + v(t,T,\omega)\ dW_t]$$

3. *Forward rate dynamics:*

$$df(t,T)(\omega) = \alpha(t,T,\omega)\ dt + \sigma(t,T,\omega)\ dW_t$$

Here $W_t$ is a standard (multidimensional) Brownian motion.

If we're given one type of dynamics, can we deduce the others? If you think about this for a while, you'd expect that bond prices and short rates are deduceable from the forward rates, and that forward rates and the short rate are deduceable from the bond prices. A model of just the short rate seems to contain too little information to deduce all bond prices and forward rates however.

Before we write down exactly how the various dynamics are related to each other, we need a stochastic Fubini Theorem and its corollary.

**Proposition 2.6** (Fubini's Theorem for Stochastic Integrals)

$$\int_0^t \int_0^T \Phi(s,S,\omega)\ dS\ dW_s(\omega) = \int_0^T \int_0^t \Phi(s,S,\omega)\ dW_s(\omega)\ dS$$

*where $(s,\omega,S) \mapsto \Phi(s,S,\omega)$ is $\mathcal{P} \times \mathcal{B}$–measurable ($\mathcal{P}$ = predictable $\sigma$–algebra, $\mathcal{B}$ = Borel algebra), and*

*(i)* $\int_0^t \Phi^2(s,S,\omega)\ ds < \infty$ *a.s. for all* $t \in [0,T]$*;*

*(ii)* $\int_0^t \left(\int_0^T \Phi(s,S,\omega)\ dS\right)^2\ ds < \infty$ *a.s. for all* $t \in [0,T]$*;*

*(iii)* $t \mapsto \int_0^T \int_0^t \Phi(s,S,\omega)\ dW_s(\omega)\ dS$ *is continuous.*

□

The proof is omitted, but may be found in Durrett, Chapter 2, Section 11.

Before we prove a corollary about the differentiation of stochastic integrals, it is convenient to gather well–known results about the differentiation of ordinary Lebesgue integrals:

**Proposition 2.7** *Assuming sufficient smoothness and regularity,*

$$\frac{\partial}{\partial x}\int_a^x f(y)\ dy = f(x)$$

$$\frac{\partial}{\partial x}\int_a^b f(x,y)\ dy = \int_a^b \frac{\partial}{\partial x} f(x,y)\ dy$$

$$\frac{\partial}{\partial x}\int_{g(x)}^{h(x)} f(x,y)\ dy = \int_{g(x)}^{h(x)} \frac{\partial}{\partial x} f(x,y)\ dy + f(x,h(x))\frac{dh}{dx} - f(x,g(x))\frac{dg}{dx}$$

□

**Corollary 2.8** (Differentiation under the integral sign)

$$\frac{\partial}{\partial T}\int_0^t v(s,T)\,dW_s = \int_0^t \frac{\partial v(s,T)}{\partial T}\,dW_s$$

**Proof:** Just like the ordinary proof of differentiation under the integral sign:

$$\begin{aligned}
\frac{\partial}{\partial T}\int_0^t v(s,T)\,dW_s &= \frac{\partial}{\partial T}\left[\int_0^t v(s,0) + \int_0^T \frac{\partial v(s,u)}{\partial T}\,du\,dW_s\right] \\
&= 0 + \frac{\partial}{\partial T}\int_0^t\int_0^T \frac{\partial v(s,u)}{\partial T}\,du\,dW_s \\
&= \frac{\partial}{\partial T}\int_0^T\int_0^t \frac{\partial v(s,u)}{\partial T}\,dW_s\,du \\
&= \int_0^t \frac{\partial v(s,T)}{\partial T}\,dW_s
\end{aligned}$$

□

Consider now the various dynamics given above, i.e. short rate, bond price and forward rate dynamics. Assume that the drifts and variance rates are $C^1$ in the $T$–variable, and sufficiently regular to allow the interchange of order of integration. Further, assume that bond prices are bounded.

The following theorem records the relationships between the various dynamics:

**Theorem 2.9** *(a) If*

$$\frac{dp}{p} = m\,dt + v\,dW_t$$

*then*

$$df = \alpha\,dt + \sigma\,dW_t$$

*where*

$$\begin{aligned}
\alpha(t,T) &= v_T(t,T)\cdot v(t,T) - m_T(t,T) \\
\sigma(t,T) &= -v_T(t,T)
\end{aligned}$$

*(b) If*

$$df = \alpha\,dt + \sigma\,dW_t$$

*then*

$$dr = a\,dt + b\,dW_t$$

*where*

$$\begin{aligned}
a(t) &= f_T(t,t) + \alpha(t,t) \\
b(t) &= \sigma(t,t)
\end{aligned}$$

*(c) If*

$$df = \alpha\,dt + \sigma\,dW_t$$

*then*

$$\frac{dp}{p} = \left[r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\right]\,dt + S(t,T)\,dW_t$$

*where*

$$A(t,T) = -\int_t^T \alpha(t,s)\, ds$$

$$S(t,T) = -\int_t^T \sigma(t,s)\, ds$$

*Here* $||\cdot||$ *is just the usual Euclidean norm.*

Before we begin the proof, note that for each $T$ we have a separate security $p(t,T)$, i.e. for each $T$ we have a separate process $(p(t,T))_{t\geq 0}$. It is to these processes that we apply Itô's formula, etc.

**Proof:** (1) $d\ln p = [m - \frac{1}{2}v^2]\, dt + v\, dW_t$ and thus

$$\ln p(t,T) = \ln p(0,T) + \int_0^t m(s,T) - \frac{1}{2}v^2(s,T)\, dt + \int_0^t v(s,T)\, dW_s$$

so that

$$-f(t,T) = \frac{\partial \ln p(t,T)}{\partial T} = \frac{\partial \ln p(0,T)}{\partial T} + \int_0^t m_T - v_T \cdot v\, ds + \int_0^t v_T\, dW_s$$

Taking differentials yields the result.

(2)

$$\begin{aligned} r(t) = f(t,t) = f(0,t) &+ \int_0^t \alpha(s,t)\, ds + \int_0^t \sigma(s,t)\, dW_s \quad \text{where} \\ \alpha(s,t) &= \alpha(s,s) + \int_s^t \alpha_T(s,u)\, du \\ \sigma(s,t) &= \sigma(s,s) + \int_s^t \sigma_T(s,u)\, du \end{aligned}$$

and thus

$$\begin{aligned} r(t) &= f(0,t) + \int_0^t \alpha(s,s)\, ds + \int_0^t\int_s^t \alpha_T(s,u)\, du\, ds + \int_0^t \sigma(s,s)\, dW_s + \int_0^t\int_s^t \sigma_T(s,u)\, du\, dW_s \\ &= f(0,t) + \int_0^t \alpha(s,s)\, ds + \int_0^t\int_0^u \alpha_T(u,s)\, ds\, du + \int_0^t \sigma(s,s)\, dW_s + \int_0^t\int_0^u \sigma_T(u,s)\, dW_s\, du \end{aligned}$$

by the stochastic Fubini theorem. Thus

$$\begin{aligned} dr(t) &= \left[\alpha(t,t) + \int_0^t \alpha_T(s,t)\, ds + \int_0^t \sigma_T(s,t)\, dW_s\right]\, dt + \sigma(t,t)\, dW_t \\ &= [\alpha(t,t) + f_T(t,t)]\, dt + \sigma(t,t)\, dW_t \end{aligned}$$

as required

(3) First define $Y(t,T) = -\int_t^T f(t,s)\, ds$, so that $p(t,T) = e^{Y(t,T)}$. Now

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)\, du + \int_0^t \sigma(u,s)\, dW_s$$

and hence

$$\begin{aligned}
Y(t,T) &= -\int_t^T f(0,s)\,ds - \int_t^T \int_0^t \alpha(u,s)\,du\,ds - \int_t^T \int_0^t \sigma(u,s)\,dW_u\,ds \\
&= -\int_t^T f(0,s)\,ds - \int_0^t \int_t^T \alpha(u,s)\,ds\,du - \int_0^t \int_t^T \sigma(u,s)\,ds\,dW_u \\
&= \left[-\int_0^T f(0,s)\,ds + \int_0^t f(0,s)\,ds\right] \\
&\qquad + \left[-\int_0^t \int_u^T \alpha(u,s)\,ds\,du + \int_0^t \int_u^t \alpha(u,s)\,ds\,du\right] \\
&\qquad + \left[-\int_0^t \int_u^T \sigma(u,s)\,ds\,dW_u + \int_0^t \int_u^t \sigma(u,s)\,ds\,dW_u\right] \\
&= Y(0,T) - \int_0^t \int_u^T \alpha(u,s)\,ds\,du - \int_0^t \int_u^T \sigma(u,s)\,ds\,dW_u \\
&\qquad + \left[\int_0^t f(0,s)\,ds + \int_0^t \int_u^t \alpha(u,s)\,ds\,du + \int_0^t \int_u^t \sigma(u,s)\,ds\,dW_u\right] \\
&= Y(0,T) - \int_0^t \int_u^T \alpha(u,s)\,ds\,du - \int_0^t \int_u^T \sigma(u,s)\,ds\,dW_u + \int_0^t f(s,s)\,ds
\end{aligned}$$

Hence

$$Y(t,T) = Y(0,T) + \int_0^t r(s)\,ds - \int_0^t \int_u^T \alpha(u,s)\,ds\,du - \int_0^t \int_u^T \sigma(u,s)\,ds\,dW_u$$

so that

$$\begin{aligned}
dY(t,T) &= \left[r(t) - \int_t^T \alpha(t,s)\,ds\right]\,dt - \left[\int_t^T \sigma(t,s)\,ds\right]\,dW_t \\
&= [r(t) + A(t,T)]\,dt + S(t,T)\,dW_t
\end{aligned}$$

and thus

$$dp = d(e^Y) = e^Y[dY + \frac{1}{2}d[Y]]$$

implies

$$\frac{dP(t,T)}{p(t,T)} = \left[r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\right]\,dt + S(t,T)\,dW_t$$

as required.

□

**Example 2.10** Synthetic Money Market Account
In a bond market, subject to the conditions enumerated before, it is possible to synthetically create a locally risk–free bank account. This is accomplished by rolling over just maturing bonds.

Consider a portfolio $V$ which, at any time, consists solely of bonds maturing at time $t+dt$. Suppose that there are $n_t$ such bonds in the portfolio, so that

$$V_t = n_t p(t, t+dt)$$

By the self–financing condition,

$$\begin{aligned}dV_t &= n_t\ dp(t,t+dt)\\ &= n_t p(t,t+dt)\left\{\left[r(t)+A(t,t+dt)+\frac{1}{2}||S(t,t+dt)||^2\right]\ dt + S(t,t+dt)\ dW_t\right\}\end{aligned}$$

Now as $dt \to 0$, also $A(t,t+dt) = -\int_t^{t+dt}\alpha(t,s)\ ds \to 0$, and $S(t,t+dt) = -\int_t^{t+dt}\sigma(t,s)\ ds \to 0$. Thus in the limit,

$$dV_t = r(t)V_t\ dt$$

which are just the dynamics of the MMA.

(Note, however, that the above argument is heuristic in nature: It requires, in any time interval, however short, the use of infinitely many types of securities.)

□

In the riskneutral world, discounted bond price processes are martingales, and thus

$$\begin{aligned}p(t,T) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\ ds}p(T,T)|\mathcal{F}_t\right]\\ &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\ ds}|\mathcal{F}_t\right]\end{aligned}$$

In a Brownian world, any equivalent measure is obtained from the objective measure by a Girsanov transformation — a consequence of the Martingale Representation Theorem. If

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\int_0^T u\ dW_t - \frac{1}{2}\int_0^T ||u||^2\ dt}$$

and if $L(t) = \mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]$ is the associated likelihood process, then $dL_t = u_t L_t\ dW_t$. Now if $\hat{p}(t,T) = \frac{p(t,T)}{A_t}$, then (under $\mathbb{Q}$)

$$d\hat{p}(t,T) = \hat{p}(t,T)v(t,T)\ d\hat{W}_t$$

(where $\hat{W}_t$ is a $\mathbb{Q}$–Brownian motion), so that

$$dp(t,T) = r(t)p(t,T)\ dt + p(t,T)v(t,T)\ d\hat{W}_t$$

Hence under $\mathbb{P}$, we have dynamics

$$\frac{dp(t,T)}{p(t,T)} = [r(t) - u(t)v(t,T)]\ dt + v(t,T)\ dW_t$$

i.e. in a Brownian world bond price dynamics are necessarily of the form $dp = pm\ dt + pv\ dW_t$.

# 3 Modelling the Short Rate

Short rate models are bond market models where the only explanatory variable is the short rate $r$. This was the earliest approach to bond market models, dating back to the paper by vasiček (1977), but short rate models have limited power. Nevertheless, principal component

analysis shows that typically 80 – 90% of price variation in the bond market can be explained by a single factor, so these models are not wholly devoid of realism.

When we specify only the short rate, the *only* exogenously given asset is the MMA $A_t$. Zero coupon bonds will be regarded not as primitive securities, but as derivatives of the short rate.

**Question:** Are bond prices uniquely determined by the $\mathbb{P}$–dynamics of the short rate?

We assume that we live in a Brownian world governed by an objective probability measure $\mathbb{P}$, with change driven by a (multidimensional) Brownian motion $W_t$. We further assume short rate dynamics of the form

$$dr(t) = \mu(t, r)\ dt + \sigma(t, r)\ dW_t$$

i.e. the short rate is an Itô diffusion.

The answer to the above question is **No!**

- The above bond market is clearly incomplete:
  - We are able to execute trading strategies which consist of putting all our money in the bank account only. This clearly doesn't give us enough freedom to replicate all possible $\mathcal{F}_T$–measurable ransom variables.
  - There is at least one source of randomness, but there are no risky assets.
  - Under any measure , the discounted MMA $\frac{A_t}{A_t}$ is a martingale. hence any measure equivalent to $\mathbb{P}$, including $\mathbb{P}$ itself, is an EMM. The EMM is not unique.
- If $\mathbb{Q} \sim \mathbb{P}$ is any equivalent measure, then $\mathbb{Q}$ generates an arbitrage-free bond market with prices
$$p(t, T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\ ds} | \mathcal{F}_t\right]$$
- In the Black–Scholes model, we also had one source of uncertainty, but there option prices are determined by the dynamics of an underlying which is traded. The crucial difference here is that the underlying is the short rate, which is not a traded security.

Nevertheless, bonds of different maturities must satisfy certain internal consistency conditions in order to exclude arbitrage. For example, if $T_1 < T_2$, then $p(t, T_1) \geq p(t, T_2)$, or else there will be arbitrage (assuming positive rates).

If we have $d$ sources of noise (i.e. $W_t$ is a $d$–dimensional Brownian motion), then we may pick $d$ maturities, and regard the bonds of those maturities as "primitive" securities; bonds of all other maturities will be "derivative". Our market now has as many risky primitive assets as securities, and is therefore complete.

## 3.1 The Term Structure PDE

Assume that we have an arbitrage–free bond market, with $\mathbb{P}$–short rate dynamics given by

$$dr(t) = \mu(t, r)\ dt + \sigma(t, r)\ dW_t$$

where $W_t$ is a *one–dimensional* $\mathbb{P}$–Brownian motion. We restrict to one dimension purely for ease of exposition – similar results hold in the multidimensional case.

Also assume that the price of a $T$–bond at time $t$ is given by a sufficiently smooth and regular function $F$:

$$p(t,T) = F(t, r(t); T) = F^T(t,r)$$

By taking two bonds of different maturities $S$ and $T$, we are able to create a locally riskless portfolio. Arbitrage considerations then dictate that the drift of this portfolio is equal to the short rate. As usual, this yields a PDE, as we now show.

First note that by Itô's formula

$$dF^T = \left[F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T\right]\, dt + \sigma F_r^T \, dW_t$$

(where subscripts denote partial derivatives), so that

$$\frac{dF^T}{F^T} = \alpha^T \, dt + \sigma^T \, dW_t$$

where

$$\alpha^T(t,r) = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T}$$

$$\sigma^T(t,r) = \frac{\sigma F_r^T}{F^T}$$

Consider now a portfolio $V$ consisting of $S$– and $T$–bonds with relative weights $w^S, w^T$ respectively. Then

$$\frac{dV}{V} = w^S \frac{dF^S}{F^S} + w^T \frac{dF^T}{F^T}$$

To eliminate risk, set $w^S\sigma^S + w^T\sigma^T = 0$. Since weights add up to 1, we therefore obtain

$$w^S = \frac{\sigma^T}{\sigma^T - \sigma^S} \qquad w^T = -\frac{\sigma^S}{\sigma^T - \sigma^S}$$

Then

$$\frac{dV}{V} = \frac{\alpha^S\sigma^T - \alpha^T\sigma^S}{\sigma^T - \sigma^S}\, dt$$

i.e.

$$\frac{\alpha^S\sigma^T - \alpha^T\sigma^S}{\sigma^T - \sigma^S} = r$$

i.e.

$$\frac{\alpha^S(t,r) - r}{\sigma^S} = \frac{\alpha^T(t,r) - r}{\sigma^T}$$

Now $\alpha^T$ is just the drift of the bond price $p(t,T) = F^T(t,r)$, and $\sigma^T$ is its volatility. Thus

$$\frac{\alpha^T(t,r) - r}{\sigma^T} = \text{ Market Price of Risk} = \lambda$$

i.e. all bonds have the same market price of risk $\lambda = \lambda(t,r)$. $\lambda$ is independent of maturity (though it may vary over time).

**Proposition 3.1** (Term Structure PDE)
*In an arbitrage–free one–factor short rate model $dr = \mu\, dt + \sigma\, dW_t$ there is a process $\lambda(t,r)$ such that*

$$\frac{\alpha^T(t,r) - r}{\sigma^T} = \text{Market Price of Risk} = \lambda$$

*Hence all bonds satisfy the following PDE*

$$\boxed{\begin{aligned} F_t^T + (\mu - \lambda\sigma)F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0 \\ F^T(T,r) &= 1 \end{aligned}}$$

**Proof:** Since

$$\begin{aligned} \alpha^T(t,r) &= \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T} \\ \sigma^T(t,r) &= \frac{\sigma F_r^T}{F^T} \end{aligned}$$

we have

$$\frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T}{\sigma F_r^T} = \lambda$$

which can easily be manipulated to yield the term structue PDE.

□

Using the Feynman–Kač formula, we see that the bond prices are given by

$$\begin{aligned} F^T(t,r) = F(t,r;T) &= \mathbb{E}_{\mathbb{Q}^\lambda}^{t,r}\left[e^{-\int_t^T r(s)\,ds}\right] \quad \text{i.e.} \\ p(t,T) &= \mathbb{E}_{\mathbb{Q}^\lambda}^{t,r}\left[e^{-\int_t^T r(s)\,ds} p(T,T)\right] \end{aligned}$$

where

$$\begin{aligned} dr &= (\mu - \sigma\lambda)\, ds + \sigma\, d\hat{W}_s \quad s \geq t \\ r(t) &= r \end{aligned}$$

are the dynamics of $r$ under $\mathbb{Q}^\lambda$. Note that, since $r_t$ is a Markov process, we have

$$\mathbb{E}_{\mathbb{Q}^\lambda}^{t,r}\left[e^{-\int_t^T r(s)\,ds} p(T,T)\right] = \mathbb{E}_{\mathbb{Q}^\lambda}\left[e^{-\int_t^T r(s)\,ds} p(T,T)|\mathcal{F}_t\right]$$

From the fact that

$$p(t,T) = \mathbb{E}_{\mathbb{Q}^\lambda}\left[e^{-\int_t^T r(s)\,ds} p(T,T)|\mathcal{F}_t\right]$$

it follows that each $\mathbb{Q}^\lambda$ is a risk–neutral measure (i.e. an EMM for the MMA).

We can also get the risk–neutral short rate dynamics from Girsanov's Theorem: a Girsanov transformation which effects the change of measure from real–world to risk–neutral has a Girsanov kernel equal to the negative of the market price of risk. Thus $-\sigma\lambda$ is added to the drift when we change the measure. Each market price of risk process $\lambda$ gives a different risk–neutral measure $\mathbb{Q}^\lambda$.

To summarize:

- In an arbitrage–free short rate model, all bonds have the same market price of risk, regardless of maturity.

- Different market prices of risk yield different risk–neutral measures — The bond market is not complete.
- The agents in the market will (implicitly) determine $\lambda$ and thus $\mathbb{Q}^\lambda$.

## 3.2 Martingale Models of the Short Rate

We model the short rate directly under a fixed riskneutral measure $\mathbb{Q}$. This is the EMM chosen by market participants, and should, in principle, be hidden in the term structure of bond prices. By calibrating a short rate model to bond prices, the market price of risk, and thus the market EMM, can be determined. This procedure is known as *inverting the yield curve*, and works as follows:

(1) Choose a short rate model (Ho–Lee, Vasiček, Cox–Ingersoll–Ross, Black–Derman–Toy) involving one or more parameters $\alpha = (\alpha_1, \dots, \alpha_n)$. The $\mathbb{Q}$–dynamics of the short rate are given by

$$dr(t) = \mu(t, r(t); \alpha)\ dt + \sigma(t, r(t); \alpha) dW_t$$

(2) Solve the term structure PDE. In the risk–neutral world, the market price of risk is $\lambda = 0$, and thus the PDE is

$$\begin{aligned} F^T_t + \mu F^T_r + \frac{1}{2}\sigma^2 F^T_{rr} - rF^T &= 0 \\ F^T(T, r) &= 1 \end{aligned}$$

for all maturities $T$. This yields theoretical bond prices

$$p(t, T; \alpha) = F^T(t, r_t; \alpha)$$

(3) Go to the market, and "observe" the empirical term structure of bond prices $\{p^*(0, T) : T \geq 0\}$.

(4) Choose $\alpha$ so that the theoretical prices $p(0, T; \alpha)$ fit the empirical prices $p^*(0, T)$ "as closely as possible" (where "close" must be defined somehow. For example, one method would be to pick maturities $T_1, \dots, T_n$ and to pick $\alpha_1, \dots, \alpha_n$ so that

$$\sum_{k=1}^{n} (p(0, T; \alpha) - p^*(0, T))^2$$

is minimized.) Let $\alpha^*$ be this "best" parameter.

(5) We now have dynamics

$$dr(t) = \mu(t, r(t); \alpha^*)\ dt + \sigma(t, r(t); \alpha^*)\ dW_t$$

under the risk–neutral measure. We can also, in principle, observe the real–world dynamics

$$dr(t) = \bar{\mu}\ dt + \bar{\sigma}\ d\bar{W}_t$$

Since $\mu = \bar{\mu} - \sigma\lambda$, we now know the market price of risk $\lambda$, and thus $\mathbb{Q} = \mathbb{Q}^\lambda$.

(6) Ideally, we would like to have

$$p(0,T;\alpha^*) = p^*(0,T) \qquad \text{for all } T$$

However, these are infinitely equations (one for each $T$), in only finitely many unknowns (the $\alpha_1, \ldots, \alpha_n$). This system is over–determined, and the model can not be made to fit the initial term structure of bond prices.

(7) However, if we choose $\alpha$ to be an infinite dimensional vector, rather than a finite dimensional one, there may be sufficient room to fit the term structure exactly. For example, the Ho–Lee model is given by

$$dr(t) = \theta(t)\,dt + \sigma\,dW_t$$

where $\sigma$ is a constant, and $W_t$ a one–dimensional Brownian motion. here $\alpha = (\theta(t) : t \geq 0)$ is an infinite –dimensional vector. The Ho–Lee model *can* be fitted to the empirically observed term structure, but this is not obvious *a priori.*

(8) Once we've parametrized our model,we can fit other interest rate derivatives.

## 3.3 Common Short Rate Models

The following are common short rate models with just one source of noise:

- Vasiček:

  $$dr = (b - ar)\,dt + \sigma\,dW_t$$

  where $a, b, \sigma$ are constants.

- Cox–Ingersoll–Ross:

  $$dr = (a - br)\,dt + \sigma\sqrt{r}\,dW_t$$

  where $a, b, \sigma$ are constants.

- Dothan or Rendlemann–Barter:

  $$dr = ar\,dt + \sigma r\,dW_t$$

  where $a, \sigma$ are constants.

- Merton:

  $$dr = a\,dt + \sigma\,dW_t$$

  where $a, \sigma$ are constants.

- Ho–Lee:

  $$dr = \theta(t)dt + \sigma\,dW_t$$

  where $\sigma$ are constants.

- Hull–White (extended Vasiček):

  $$dr = (b(t) - a(t)r)\,dt + \sigma(t)\,dW_t$$

- Hull–White (extended CIR):

$$dr = (b(t) - a(t)r)\ dt + \sigma(t)\sqrt{r}\ dW_t$$

- Black–Derman–Toy:

$$dr = a(t)r\ dt + \sigma(t)r\ dW_t$$

- Black–Karasinski:

$$dr = (a(t)r + b(t)r\ln r)\ dt + \sigma(t)r\ dW_t$$

All of the above can be written as

$$dr = (\alpha_1(t) + \alpha_2(t)r + \alpha_3(t)r\ln r)\ dt + (\beta_1(t) + \beta_2(t)r)^\nu\ dW_t$$

## 3.4 Term Structure Derivatives

Consider the general short rate model $dr(t) = \mu(t,r)\ dt + \sigma(t,r)\ dW_t$. Suppose that an interest rate derivative has a terminal payoff $\Phi(T, r_T)$ and a dividend rate $q(t, r_t)$ over the interval $[0, T]$. The time–$t$ price of the derivative is obtained via an arbitrage argument: Start with a portfolio $V$ consisting of one derivative $F$ and $-n$ $T$–bonds $p$. Because of the dividends, we obtain

$$dV = dF - n\ dp + q\ dt$$

But choosing $n = \frac{\partial F}{\partial r} / \frac{\partial p}{\partial r}$ will make the portfolio locally riskless, and we obtain

$$\frac{F_t + \frac{1}{2}\sigma^2 F_{rr} - rF + q}{\frac{\partial F}{\partial r}} = \frac{P_t + \frac{1}{2}\sigma^2 P_{rr} - rP}{\frac{\partial P}{\partial r}}$$

Now the term structure PDE states that

$$P_t + \frac{1}{2}\sigma^2 P_{rr} - rP = -(\mu - \sigma\lambda)P_r$$

and thus

$$\begin{cases} F_t + (\mu - \sigma\lambda)F_r + \dfrac{1}{2}\sigma^2 F_{rr} - rF + q = 0 \\ \hfill F(t, r_T) = \Phi(T, r_T) \end{cases}$$

This is the generalized term structure equation for an interest rate derivative $F$ (where $\lambda = 0$ if we model the short rate in the risk–neutral world).

The value of the interest rate derivative is clearly

$$F(t, r_t; T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_u\ du}\Phi(T, r_T) + \int_t^T e^{-\int_t^s r_u\ du} q(s, r_s)\ ds\right]$$

A trivial generalization of the Feynman–Kac argument shows that the solution of a PDE of the form

$$F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} - rF + h = 0 \qquad F(T, x) = \Phi(T, x)$$

is given by

$$F(t, x) = \mathbb{E}^{r,x}\left[\int_t^T e^{-\int_t^u r_s\ ds} h(u, X_u)\ du + e^{-\int_t^T r_s\ ds}\Phi(T, X_T)\right]$$

where

$$dX_s = \mu\ ds + \sigma\ dW_s \quad \text{for } t \le s \le T \qquad \text{and} \qquad X_t = x$$

**Example 3.2** (a) A call with strike $K$ and expiry $\tau$ on a discount bond $p(t,T)$ (with $T > \tau$) has $q = 0$ and $\Phi(\tau, r) = (p(\tau, T) - K)^+$. To calculate the option price, we first have to solve the term structure PDE to get the bond prices, and then once more to price the option.

(b) An interest swap (pay–fixed) can be idealized as a contract paying a divided rate $h(t, r_t) = r_t - r^*$, where $r^*$ is the agreed-upon fixed rate (the swap rate at inception). Here $\Phi(t, r_T) = 0$, and so

$$F(t, r_t) = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u\,du}(r_s - r^*)\,ds\right]$$

Now a floating rate note paying a continuous rate $r_t$ must be priced at par $= 1$ in order to avoid arbitrage (Why?), and thus

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du} + \int_t^T e^{-\int_t^s r_u\,du} r_s\,ds\right] = 1$$

It follows that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u\,du} r_s\,ds\right] = 1 - \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du}\right] = 1 - p(t,T)$$

Hence the value of this idealized swap is

$$F(t, r_t) = 1 - p(t,T) - r^* \int_t^T p(t,s)\,ds$$

The swap rate at time $t$ for maturity $T$ sets the value of the swap to zero, and is

$$r^*(t,T) = \frac{1 - p(t,T)}{\int_t^T p(t,s)\,ds}$$

(c) A cap can be idealized as a derivative with zero terminal payoff and a dividend rate $q(t, r_t) = (r_t - \hat{r})^+$, where $\hat{r}$ is the cap rate.

□

We will spend quite a bit of effort pricing options on discount bonds in the next few pages. But what about coupon –bearing bonds, which are, after all, more commonly traded in the market? *Jamshidian's Trick* sometimes holds the answer: In a short rate model, a call on a coupon bearing bond can be priced as a portfolio of calls on zero coupon bonds, provided that the value $p(t,T) = p(t, r_t; T)$ of the zero coupon bonds is a strictly decreasing function of the short rate.

**Theorem 3.3** *Let $\mathcal{C}^{K,\tau}(t, r_t)$ be the time–$t$ value of a call on a coupon bond $\mathcal{B}$, where $\tau$ is the expiry of the call, and $K$ the strike. Suppose that the coupon bond pays a coupon $Y_i$ at date $T_i$, where $\tau < T_1 < \cdots < T_N$. Let*

$$K^i = p(\tau, r^*, T_i)$$

*where $r^*$ solves*

$$\mathcal{B}(\tau, r^*) = K$$

Recall that $\mathcal{B}(\tau,r)=\sum_i Y_i p(\tau,r,T_i)$. Since each $p(t,r,T_i)$ is a decreasing function of $r$, so is $\mathcal{B}(t,r)$, which implies that $r^*$ is unique. Solve numerically for $r^*$, e.g. via bisection method. *Then*

$$\mathcal{C}^{K,\tau}(t,r)=\sum_i Y_i C^{K_i,\tau,T_i}(t,r)$$

*where $C^{K,T,S}$ is the time–$t$ value of a strike $K$, expiry $T$ call on a zero coupon bond $p(t,S)$ (with $S\geq T$).*

**Proof:** The payoff of the call on $\mathcal{B}$ is

$$(\mathcal{B}(\tau,r_\tau)-K)^+=\left(\sum_i Y_i p(\tau,r_\tau,T_i)-K\right)^+$$

Since each $p(t,r,T)$ is decreasing in $r$, so is $\mathcal{B}(\tau,r_\tau)$. Let $r^*$ be the unique value of $r_\tau$ for which the call $\mathcal{C}$ expires at the money, i.e. for which

$$\mathcal{B}(\tau,r^*)=K$$

Now define $K_i=p(\tau,r^*,T_i)$. Then

$$\sum_i Y_i K_i=K$$

Now consider two cases:

Case 1: If $r_\tau<r^*$, then

$$\sum_i Y_i p(\tau,r_\tau,T_i)>\sum_i Y_i p(\tau,r^*,T_i)=K$$

and

$$p(\tau,r_\tau,T_i)>p(\tau,r^*,T_i)=K_i$$

Thus if $\mathcal{C}^{K,\tau}$ expires in the money, then so does each $C^{K_i,\tau,T_i}$, and

$$\begin{aligned}\left(\sum_i Y_i p(\tau,r_\tau,T_i)-K\right)^+&=\sum_i Y_i p(\tau,r_\tau,T_i)-K\\&=\sum_i Y_i\left(p(\tau,r_\tau,T_i)-K_i\right)\\&=\sum_i Y_i\left(p(\tau,r_\tau,T_i)-K_i\right)^+\end{aligned}$$

Case 2: If $r_\tau\geq r^*$, then $\sum_i Y_i p(\tau,r_\tau,T_i)\leq K$ and $p(\tau,r_\tau,T_i)\leq K_i$. Thus if $\mathcal{C}^{K,\tau}$ expires out of the money, then so does each $C^{K_i,\tau,T_i}$, so

$$\begin{aligned}0&=\left(\sum_i Y_i p(\tau,r_\tau,T_i)-K\right)^+\\&=\sum_i Y_i\left(p(\tau,r_\tau,T_i)-K_i\right)^+\end{aligned}$$

Hence in either case

$$\mathcal{C}^{K,\tau}(\tau, r_\tau) = \sum_i Y_i C^{K_i,\tau,T_i}(\tau, r_\tau)$$

Thus, by the law of one price,

$$\mathcal{C}^{K,t}(t, r_t) = \sum_i Y_i C^{K_i,t,T_i}(t, r_t)$$

for all $t \leq \tau$ as well.

$\dashv$

### 3.5 Lognormal Models

The Dothan, Rendleman–Barter, Black–Derman–Toy and Black–Karasinski all yield lognormal short rate dynamics. All suffer from the following problem: Let $A_t$ denote the money market account, with $dA_t = r_t A_t \, dt, A_0 = 1$. Then

$$\mathbb{E}\left[e^{-\int_0^t r_s \, ds}\right] \approx \mathbb{E}\left[e^{\frac{t}{2}(r_0 + r_t)}\right]$$

for sufficiently small $t$. Now, since $r_t$ is lognormal, define $Y_t = \ln r_t$. Then we have an expectation of the form

$$\mathbb{E}\left[e^{\frac{t}{2}e^{Y_t}}\right] = \mathbb{E}\left[e^{e^Z}\right]$$

for some normally distributed $Z$. Now

$$\int_{-\infty}^{\infty} e^{e^z} e^{-z^2/2} \, dz = \infty$$

as $e^{e^z} >> e^{-z^2/2}$ for reasonable values of $z$. Hence $\mathbb{E}[A(t)] = \infty$ even if $t$ is small, i.e. the bank account, on average, explodes.

Indeed, it can be shown that

$$\mathbb{E}\left[\frac{1}{p(t,T)}\right] = \infty \qquad \text{for all } t > 0$$

One consequence of this *lognormal explosion* is that one cannot price *Eurodollar futures.*

# 4 Affine Term Structure Models

## 4.1 Mechanics of ATS models

**Definition 4.1** A short rate model is said to possess *affine term structure* (ATS) if bond prices are given by

$$p(t,T) = F^T(t, r(t)) = e^{A(t,T) - B(t,T) r(t)}$$

where $A(t,T), B(t,T)$ are (sufficiently regular) deterministic functions.

$\square$

Note that not all short rate models are affine term structure models. However, the class of affine term structure models is quite well understood: They are those for which both the drift and the volatility–squared are affine functions of the short rate.

For consider a short rate model with *risk–neutral* dynamics $dr(t) = \mu(t,r)\,dt + \sigma(t,r)\,dW_t$ and suppose that bond prices are of the form $p(t,T) = F^T(t,r(t)) = e^{A(t,T)-B(t,T)r(t)}$. Substituting this expression into the term structure PDE

$$\begin{aligned} F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0 \\ F^T(T,r) &= 1 \end{aligned}$$

we obtain

$$A_t - \mu B + \frac{1}{2}\sigma^2 B^2 - (1+B_t)r = 0$$

Moreover, since $p(T,T) = 1$, we must have $A(T,T) = 0 = B(T,T)$.

If we assume that the drift and volatility of the short rate can be expressed in the form

$$\begin{aligned} \mu(t,r) &= \alpha(t)r + \beta(t) \\ \sigma^2(t,r) &= \gamma(t)r + \delta(t) \end{aligned}$$

then we obtain

$$\left[A_t - \beta B + \frac{1}{2}\delta B^2\right] = \left[1 + B_t + \alpha B - \frac{1}{2}\gamma B^2\right] r$$

The lefthand side is independent of $r$, whereas the righthand side contains $r$. This can happen only if both sides are identically zero, so that we obtain a coupled system of differential equations:

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \end{cases}$$

$$\begin{cases} B_t(t,T) = -\alpha(t)B(t,T) + \frac{1}{2}\gamma(t)B^2(t,T) - 1 \\ B(T,T) = 0 \end{cases}$$

Note that the bottom equation (a *Riccatti equation*) does not contain $A$, and can therefore be solved (in principle, although this may be quite hard). The solution can then be plugged into the top equation to solve for $A$. To solve this equation, simply integrate both sides (from $t$ to $T$).

Thus a short rate model has affine term structure whenever $\mu, \sigma$ are of the form $\mu(t,r) = \alpha(t)r + \beta(t)$ and $\sigma^2(t,r) = \gamma(t)r + \delta(t)$. The Ho–Lee, Cox–Ingersoll–Ross, Merton, Vasiček and Hull–White models all have ATS. The Dothan and Black–Derman–Toy models do not.

**Example 4.2** The Vasiček Model
Here we have $dr_t = (b - ar)\;dt + \sigma\;dW_t$, where $a, b, \sigma$ are constants. Thus we have $\alpha = -a, \beta = b, \gamma = 0, \delta = \sigma^2$, all constant.
The system of differential equations that must be solved is therefore

$$\begin{cases} A_t = bB - \frac{1}{2}\sigma^2 B^2 \\ A(T,T) = 0 \end{cases}$$

$$\begin{cases} B_t = aB - 1 \\ B(T,T) = 0 \end{cases}$$

The bottom equation is a first order linear equation. This can easily be solved: Use $e^{-at}$ as an integrating factor to obtain

$$B(t,T) = e^{at}\left[\frac{1}{a}e^{-at} + C(T)\right]$$

and then use $B(T,T) = 0$ to get $C(T) = -\frac{1}{a}e^{-aT}$. Hence

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

Plug this into the equation for $A$ to obtain

$$\begin{cases} A_t = bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ \quad = b\frac{1}{a}\left[1 - e^{-a(T-t)}\right] - \frac{1}{2}\sigma^2\frac{1}{a^2}\left[1 - e^{-a(T-t)}\right]^2 \\ A(T,T) = 0 \end{cases}$$

Integrate both sides:

$$\begin{aligned} A(t,T) &= A(T,T) - \int_t^T A_t(s,T)\,ds \\ &= \frac{1}{2}\sigma^2\int_t^T \frac{1}{a^2}\left[1 - e^{-a(T-s)}\right]^2\,ds - b\int_t^T \frac{1}{a}\left[1 - e^{-a(T-s)}\right]\,ds \\ &= -\frac{\sigma^2 B^2}{4a} + \frac{(B - (T-t))(ab - \frac{1}{2}\sigma^2)}{a^2} \end{aligned}$$

Now that $A, B$ have been found, bond prices are given by the equation $p(t,T) = e^{A(t,T)-B(t,T)r(t)}$.

□

In order to invert the yield curve in the above example, the parameters $a, b\sigma$ must now be chosen so that the model fits empirical (observed) term structure of bond prices $\{p^*(0,T) : T \geq 0\}$ as "closely" as possible. Clearly, however, we have infinitely many bond prices, but only three parameters, i.e. the system is highly over-determined, and therefore we cannot generally choose $a, b, \sigma$ such that $e^{A(0,T)-B(0,T)r_0} = p^*(0,T)$, i.e. the model cannot be made to fit the observed term structure exactly (unless we are astoundingly fortunate). The Vasiček model is able to fit, exactly, just 3 bonds.

**Example 4.3** Cox–Ingersoll–Ross model The risk–neutral short rate dynamics assumed are

$$dr_t = (b - ar_t)\,dt + \sigma\sqrt{r_t}\,dW_t, \qquad a, b, \sigma, r_0 > 0$$

This is mean reverting (to $b/a$). Since the volatility term $\sigma\sqrt{r_t}$ tends to zero as $r_t \to 0$ (which is consistent with observation), positive rates are assured (which is also consistent with

observation). Postulating $p(t,T) = e^{A(t,T)-B(t,T)r_t}$, we quickly determine, by substituting into the term structure PDE, that

$$\begin{aligned} B_t &= aB + \frac{1}{2}\sigma^2 B^2 - 1 \qquad && B(T,T) = 0 \\ A_t &= bB && A(T,T) = 0 \end{aligned}$$

To solve the Riccati equation for $B$, we try a solution of the form

$$B(t,T) = \frac{X(t)}{cX(t) + d}$$

Then

$$B_t = \frac{X_t}{cX + d} - \frac{cXX_t}{(cX + d)^2}$$

and hence, substituting into the equation for $B_t$, we see that

$$-dX_t + X^2(ac + \frac{1}{2}\sigma^2 - c^2) + X(ad - 2cd) - d^2 = 0 \qquad X(T) = 0$$

Choose $c$ to ensure that $a + \frac{1}{2}\sigma^2 - c^2 = 0$, i.e. $c = \frac{1}{2}(a + \sqrt{a^2 + 2\sigma^2})$. We then have a order linear differential equation

$$X_t + \kappa X = -d \qquad \text{where } \kappa = -a + 2c = \sqrt{a^2 + 2\sigma^2}$$

Since $X(T) = 0$,we see that

$$X(t) = \frac{d}{\kappa}[e^{\kappa(T-t)} - 1]$$

Hence

$$\begin{aligned} B(t,T) &= \frac{X(t)}{cX(t) + d} \\ &= \frac{e^{\kappa(T-t)} - 1}{\frac{1}{2}(\kappa + a)(e^{\kappa(T-t)} - 1) + \kappa} \\ &= \frac{2(e^{\kappa(T-t)} - 1)}{2\kappa + (a + \kappa)(e^{\kappa(T-t)} - 1)} \qquad \text{where } \kappa = \sqrt{a^2 + 2\sigma^2} \end{aligned}$$

Then $A(t,T)$ is obtained by integrating:

$$A(t,T) = A(T,T) - \int_t^T A_t(s,T)\,ds = -b\int_t^T B(s,T)\,ds$$

The solution is

$$A(t,T) = \frac{2b}{\sigma^2}\ln\left[\frac{2\kappa e^{\frac{1}{2}(a+\kappa)(T-t)}}{2\kappa + (a + \kappa)(e^{\kappa(T-t)} - 1)}\right]$$

as can be verified by differentiation.

$\square$

**Example 4.4** Ho–Lee Model
We are given risk–neutral short rate dynamics $dr(t) = \theta(t)\ dt + \sigma\ dW_t$, where $\theta(t)$ is deterministic and $\sigma$ a constant. The model has ATS with $\alpha = 0, \beta = \theta, \gamma = 0, \delta = \sigma^2$. This leads to two differential equations. The first is

$$\begin{cases} B_t = -1 \\ B(T,T) = 0 \end{cases}$$

which has solution $B(t,T) = T - t$ (as can be seen by integrating both sides from $t$ to $T$). The second DE is

$$\begin{cases} A_t = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ \quad = \theta(t)(T-t) - \frac{1}{2}\sigma^2(T-t)^2 \\ A(T,T) = 0 \end{cases}$$

Integrating both sides from $t$ to $T$ yields

$$A(t,T) = -\int_t^T \theta(s)(T-s)\ ds + \frac{1}{6}\sigma^2(T-t)^3$$

We now choose the function $\theta(t)$ so as to fit the initial term structure of bond prices $\{p^*(0,T) : T \geq 0\}$, or, equivalently, the observed term structure of (instantaneous) forward rates $\{f^*(0,T) : T \geq 0\}$.

Recall that $f^*(0,T) = -\frac{\partial \ln p^*(0,T)}{\partial T}$. With affine term structure, we have $p^*(0,T) = e^{A(0,T)-B(0,T)r_0}$, so

$$\ln p^*(0,T) = -\int_0^T \theta(s)(T-s)\ ds + \frac{1}{6}\sigma^2 T^3 - r_0 T$$

Differentiating with respect to $T$, we see that

$$f^*(0,T) = \int_0^T \theta(s)\ ds - \frac{1}{2}\sigma^2 T^2 + r_0$$

Differentiating once more with respect to $T$, we obtain

$$\frac{\partial f^*(0,T)}{\partial T} = \theta(T) - \sigma^2 T$$

and thus we have found $\theta$:

$$\theta(t) = f_T^*(0,t) + \sigma^2 t$$

We can use this to calculate $A(t,T)$:

$$\begin{aligned} A(t,T) &= \int_t^T (f_T^*(0,s) + \sigma^2 s)(s-T)\ ds + \frac{1}{6}\sigma^2(T-t)^3 \\ &= f^*(0,s)(s-T)|_t^T - \int_t^T f^*(0,s)\ ds + \sigma^2\left[\frac{s^3}{3} - \frac{s^2 T}{2}\right]_t^T + \frac{1}{6}\sigma^2(T-t)^3 \\ &= f^*(0,t)(T-t) + \int_t^T \frac{\partial \ln p^*(0,s)}{\partial T}\ ds - \frac{1}{2}\sigma^2 t(T-t)^2 \\ &= f^*(0,t)(T-t) + \ln\left(\frac{p^*(0,T)}{p^*(0,t)}\right) - \frac{1}{2}\sigma^2 t(T-t)^2 \end{aligned}$$

Using the fact that the Ho–Lee model has ATS, we see that bond prices are given by

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\left( f^*(0,t)(T-t) - \frac{1}{2}\sigma^2 t(T-t)^2 - (T-t)r(t) \right)$$

where

$$\begin{aligned} r(t) &= r_0 + \int_0^t \theta(s)\, ds + \int_0^t \sigma\, dW_s \\ &= r_0 + f^*(0,t) - f^*(0,0) + \frac{1}{2}\sigma^2 t^2 + \sigma W_t \\ &= f^*(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma W_t \end{aligned}$$

because $r_0 = f^*(0,0)$. It follows that $\mathbb{E}[r_t] \to \infty$ (under the riskneutral measure). This is clearly a flaw in the model.

Since the short rate is Gaussian, future bond prices are lognormally distributed under the risk–neutral measure. In particular, there is a non–zero probability that a bond will, at some future date, trade above par (i.e. that interest rates become negative). This is clearly another flaw in the model.

Now that we've calculated the evolution of future bond prices and rates, let's have a look at future forward rates. Since $f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T}$, we see that

$$\begin{aligned} f(t,T) &= f^*(0,T) - f^*(0,t) + \sigma^2 t(T-t) + r_t \\ &= f^*(0,T) + \sigma^2 t(T - \frac{1}{2}t) + \sigma W_t \end{aligned}$$

using the expression for $r_t$ obtained earlier. Note that $f(t,t) = r_t$.

Now if we fix $t > 0$, we see that $\mathbb{E}[f(t,T)] \to \infty$ as $T \to \infty$. Indeed, for large values of $T$, $f(t,T) \approx kT$. Thus even if the initial forward curve is bounded above, it will be unbounded an instant later. This is another flaw in the Ho–Lee model.

□

**Example 4.5** The Hull–White (extended Vasiček) Model
Consider the short rate model with risk–neutral dynamics

$$dr_t = (b(t) - ar_t)\, dt + \sigma\, dW_t$$

where $b(t)$ is deterministic, $a, \sigma$ are constants and $W_t$ is a one–dimensional Brownian motion. This is clearly an affine term structure model $dr_t = (\alpha(t)r_t + \beta(t))\, dt + \sqrt{\gamma(t)r_t + \delta(t)}\, dW_t$, with $\alpha(t) = -a, \beta(t) = b, \gamma(t) = 0$ and $\delta(t) = \sigma^2$. Substituting $p(t,T) = e^{A(t,T) - B(t,T)r_t}$ into the term structure PDE yields

$$\begin{aligned} B_t(t,T) &= aB(t,T) - 1 \\ B(T,T) &= 0 \end{aligned}$$

and

$$\begin{aligned} A_t(t,T) &= b(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) &= 0 \end{aligned}$$

Hence

$$\begin{aligned}B(t,T) &= \frac{1}{a}(1-e^{-a(T-t)})\\ A(t,T) &= \int_t^T -b(u)B(u,T) + \frac{1}{2}\sigma^2 B^2(u,T)\,du\end{aligned}$$

Fitting the initial term structure of bond prices is equivalent to fitting the initial term structure of forward rates. The latter is more convenient. Now since $f(0,T) = -\frac{\partial \ln P(0,T)}{\partial T} = -A_T(0,T) + B_T(0,T)r_0$, and since $B_T(t,T) = e^{-a(T-t)}$, we observe

$$\begin{aligned}f(0,T) &= \int_0^T b(u)B_T(u,T) + \sigma^2 B_T(u,T)B(u,T)\,du + B_T(0,T)r_0\\ &= \int_0^T b(u)e^{-a(T-u)}\,du - \frac{\sigma^2}{2a^2}(1-e^{-a(T-t)})^2 + e^{-aT}r_0\end{aligned}$$

We side–step the computation of these integrals using the following trick: Define

$$\begin{aligned}x(t) &= e^{-at}r_0 + \int_0^t b(u)e^{-a(t-u)}\,du\\ y(t) &= \frac{\sigma^2}{2a^2}(1-e^{-at})^2\end{aligned}$$

Note that

$$\begin{aligned}x'(T) &= -ar_0e^{-aT} + b(T) - a\int_0^T b(u)e^{-a(T-u)}\,du\\ &= -ax(T) + b(T)\end{aligned}$$

Now $f(0,T) = x(T) - y(T)$, and so

$$\begin{aligned}b(T) &= x' + ax\\ &= f_T(0,T) + y'(T) + ax(T)\\ &= f_T(0,T) + y'(T) = a[f(0,T) + y(T)]\end{aligned}$$

Thus, noting that $y(t) = \frac{\sigma^2}{2a^2}(1-e^{-at})^2 = \frac{1}{2}\sigma^2B^2(0,t)$ and thus that $y'(t) = \frac{\sigma^2}{a}(1-e^{-at})e^{-at} = \sigma^2B(0,t)B_T(0,t)$, we obtain

$$\begin{aligned}b(t) &= f_T^*(0,t) + \sigma^2B(0,t)B_T(0,t) + a[f^*(0,t) + \frac{1}{2}\sigma^2B^2(0,t)]\\ &= f_T^*(0,t) + af^*(0,t) + \frac{\sigma^2}{2a}[1-e^{-2at}]\end{aligned}$$

This is the function $b(t)$ which will fit forward rates to the observed term structure $\{f^*(0,T) : T \geq 0\}$.

Since we now know $b(t)$ we can calculate $A(t,T)$:

$$A(t,T) = \int_t^T -b(u)B(u,T) + \frac{1}{2}\sigma^2B^2(u,T)\,du$$

Now note that $b(u) = x'(u) + ax(u) = e^{-au}\frac{d\,xe^{au}}{du}$ so that

$$\begin{aligned}\int_t^T b(u)B(u,T)\,du &= \frac{1}{a}\int_t^T e^{-au}\frac{d\,xe^{au}}{du}(1-e^{-a(T-u)})\,du\\ &= \frac{1}{a}\int_t^T (e^{-au}-e^{-aT})\,d(xe^{au})\\ &= \frac{1}{a}\left[x(u)e^{au}(e^{-au}-e^{-aT})\right]_t^T - \frac{1}{a}\int_t^T xe^{au}\cdot -ae^{-au}\,du\\ &= -\frac{1}{a}x(t)(1-e^{-a(T-t)}) + \int_t^T x(u)\,du\\ &= -\left[f(0,t)+\frac{\sigma^2}{2}B^2(0,t)\right]B(t,T) + \int_t^T -\frac{\partial p(0,u)}{\partial T} + \frac{\sigma^2}{2}B^2(0,u)\,du\\ &= -f(0,t)B(t,T) - \ln\frac{p(0,T)}{p(0,t)} - \frac{\sigma^2}{2}B^2(0,t)B(t,T) + \int_t^T \frac{\sigma^2}{2}B^2(0,u)\,du\end{aligned}$$

Hence

$$A(t,T) = f(0,t)B(t,T) + \ln\frac{p(0,T)}{p(0,t)} + \frac{\sigma^2}{2}\left[\int_t^T B^2(u,T) - B^2(0,u)\,du + B^2(0,t)B(t,T)\right]$$

Now, after a few lines of manipulation,

$$\int_t^T B^2(u,T) - B^2(0,u)\,du + B^2(0,t)B(t,T) = -\frac{1}{2a}B^2(t,T)(1-e^{-2at})$$

as you can easily check, substituting $B(t,T) = \frac{1}{a}(1-e^{-a(T-t)})$. Thus

$$A(t,T) = f(0,t)B(t,T) + \ln\frac{p(0,T)}{p(0,t)} - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at})$$

Substituting $p(t,T) = e^{A(t,T)-B(t,T)r_t}$ we obtain:

$$p(t,T) = \frac{p(0,T)}{p(0,t)}e^{f(0,t)B(t,T)-\frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at})-B(t,T)r_t}$$

□

We have thus found the following bond prices:

**Theorem 4.6** *(a) In the Ho–Lee model, bond prices (fitted to the initial term structure) are given by*

$$p(t,T) = \frac{p(0,T)}{p(0,t)}\exp\left(f(0,t)(T-t) - \frac{1}{2}\sigma^2 t(T-t)^2 - (T-t)r(t)\right)$$

*(b) In the Hull–White (extended Vasiček) model, bond prices (fitted to the initial term structure) are given by*

$$p(t,T) = \frac{p(0,T)}{p(0,t)}\exp\left(f(0,t)B(t,T) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r_t\right)$$

*where* $B(t,T) = \frac{1}{a}(1-e^{-a(T-t)})$.

□

## 4.2 Bond Options

In the chapter on changes of numéraire, we obtained the following general option formula: The price of a call $C$ with strike $K$ and maturity $T$ on an underlying $S$ is given by

$$C_0 = S_0 \mathbb{Q}_S(S_T \geq K) - Kp(0,T)\mathbb{Q}^T(S_T \geq K)$$

where $\mathbb{Q}_S, \mathbb{Q}^T$ are the EMM's associated with numéraires $S_t, p(t,T)$ respectively.

In order to use this formula, and to get Black–Scholes type solutions to option pricing problems, we assumed that the volatility of the securities is deterministic, and then obtained

**Theorem 4.7** *If $\hat{S}_t = \frac{S_t}{p(t,T)}$ is an Itô process of the form $\frac{d\hat{S}_t}{\hat{S}_t} = \mu(t)\, dt + \sigma(t) \cdot\, dW_t$, and if $\sigma(t)$ is* deterministic*, then the value of a call $C$ with maturity strike $K$ and $T$ on underlying security $S$ is given by*

$$C_0 = S_0 N(d_1) - Kp(0,T)N(d_2)$$

*where*

$$\sigma_{av} = \sqrt{\frac{1}{T}\int_0^T ||\sigma(t)||^2\, dt}$$

$$d_1 = \frac{\ln \frac{S_0}{Kp(0,T)} + \frac{1}{2}\sigma_{av}^2 T}{\sigma_{av}\sqrt{T}} \qquad d_2 \quad = d_1 - \sigma_{av}\sqrt{T}$$

*Put–call parity yields*

$$P_0 = -S_0 N(-d_1) + Kp(0,T)N(-d_2)$$

*for the price of a corresponding put.*

□

We can now use this theorem to price bond options.

**Example 4.8** Bond Options in the Ho–Lee Model
Consider a European call option $C$ with strike $K$ and maturity $T$ on a discount bond $p(t,S)$ (where $S > T$). In the Ho–Lee model, with risk–neutral dynamics $dr_t = \theta(t)\, dt + \sigma\, dW_t$, bond prices have dynamics

$$\frac{dp(t,T)}{p(t,T)} = r\, dt - \sigma(T-t)\, dW_t$$

The drift term is $r$, because bond prices have drift $r$ under the risk–neutral measure, just like all other traded securities. The volatility is obtained from the affine term structure: $p(t,T) = e^{A(t,T)-B(t,T)r_t}$, and we found that $B(t,T) = T-t$ (and we don't care about the value of $A(t,T)$ right now.) Thus the bond volatilities are deterministic: $p(t,T)$ has volatility $-\sigma(T-t)$ and $p(t,S)$ has volatility $-\sigma(S-t)$. Now the underlying security is $p(t,S)$, and $\hat{p}(t,S) = \frac{p(t,S)}{p(t,T)}$ has deterministic (indeed, *constant*) volatility $-\sigma(S-t)+\sigma(T-t) = -\sigma(S-T)$. This is because the volatility of a ratio of two assets is just the difference of their volatilities.

It follows that $\hat{p}(t,S)$ is lognormally distributed, and that $\ln \hat{p}(t,S)$ has variance $\sigma_{av}^2 T = \int_0^T \sigma^2(T-S)^2\, dt = \sigma^2(S-T)^2 T$. It follows that the price of the call is

$$C_0 = p(0,S)N(d_1) - Kp(0,T)N(d_2)$$

where

$$d_1 = \frac{\ln \frac{p(0,S)}{Kp(0,T)} + \frac{1}{2}\sigma^2(S-T)^2 T}{\sigma(S-T)\sqrt{T}}$$
$$d_2 = d_1 - \sigma(S-T)\sqrt{T}$$

□

**Example 4.9** Bond Options in the Hull–White (extended Vasiček) Model
We tackle once more the problem of pricing a call with strike $K$ and maturity $T$ on a zero coupon bond $p(0,S)$, where $S > T$. It ought to be clear from the analysis of bond options in the Ho–Lee model that we need mainly to find the volatility of the bonds $p(t,T)$. Now, as for the Ho–Lee model, the riskneutral dynamics of $p(t,T)$ are

$$\frac{dp(t,T)}{p(t,T)} = r\,dt - B(t,T)\sigma\,dW_t$$

so that the volatility of $p(t,T)$ is $-\frac{\sigma}{a}(1-e^{-a(T-t)})$. The asset ratio $\hat{p}_t = p(t,S)/p(t,T)$ therefore has volatility $\frac{\sigma}{a}(e^{-aS}-e^{-aT})e^{at}$ at time $t$. Thus the average volatility–squared is

$$\sigma_{av}^2 T = \int_0^T \frac{\sigma^2}{a^2}(e^{-aS}-e^{-aT})^2 e^{2at}\,dt = \frac{\sigma^2}{2a^3}(1-e^{-a(S-T)})^2(1-e^{-2aT})$$

We now find that the value of the call is simply

$$p(0,S)N(d_1) - Kp(0,T)N(d_2)$$

where

$$d_1 = \frac{\ln \frac{p(0,S)}{Kp(0,T)} + \frac{1}{2}\sigma_{av}^2 T}{\sigma_{av}\sqrt{T}} \qquad d_2 = d_1 - \sigma_{av}\sqrt{T}$$

□

# 5 The Heath–Jarrow–Morton Framework

## 5.1 The Set–Up

Up till now, we have studied interest rate models in which the short rate is the only explanatory variable. Such an approach has many obvious advantages:

- Specifying $r$ as the solution of an SDE allows us to use Markov theory, which leads to PDE's (e.g., via the Feynman–Kač theorem, or the Kolmogorov forward and backward equations) that can be solved;
- If we're lucky, we can obtain analytical formulas for bond prices and bond option prices (as we did for the Ho–Lee and Hull–White (extended Vasiček) models.

However, the short rate modelling approach has some obvious disadvantages as well:

- It is unreasonable to regard the short rate as the only explanatory variable — it is difficult to incorporate views about different times in the future;

- It can be quite difficult to fit a realistic volatility structure;
- In order for the model to have even a remote chance of being correct, it is necessary to invert the yield curve (i.e. to fit the model to the initial term structure of bond prices). This can be quite difficult as well.

The Heath–Jarrow–Morton (HJM) approach circumvents some of these difficulties by specifying dynamics for the entire (uncountable) family of forward rates. For a fixed $T \geq 0$, assume that the forward rate $f(t,T)$ has "real–world" dynamics

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,d\hat{W}_t \qquad T \geq 0, 0 \leq t \leq T$$

where $\hat{W}_t$ is a *finite–dimensional* Brownian motion under the real world measure $\mathbb{P}$, and $\alpha(t,T)$ and $\sigma(t,T)$ are adapted (and sufficiently regular to ensure that most of the operations below are permissible. For example, it is often necessary to assume that $\alpha(t,T)$ is jointly measurable in the $t$– and $T$–variables.)

Thus we have infinitely many SDE's, one for each maturity $T$. Each such SDE has an initial condition, namely $f(0,T) = f^*(0,T)$, where $f^*(0,T)$ is the observed term structure. the advantage of this approach is that the initial term structure is fitted automatically — it is an initial condition! — so that inverting the yield curve becomes unnecessary. It is also easier to incorporate views about different maturities, because we have many different SDE's. (The disadvantage, of course, is that we have many, many SDE's.) These are still manageable, because we assume that the bond market is driven by finitely many sources of noise. But this leads to another difficulty:

**Remarks 5.1** Given $\alpha(t,T), \sigma(t,T)$ and $\{f^*(0,T) : T \geq 0\}$, we can solve the SDE's for the forward rate, so that we have specified the entire term structure $\{f^*(t,T) : T \geq 0, 0 \leq t \leq T\}$ at all times and all maturities, and thus the entire term structure of bond prices

$$p(t,T) = e^{-\int_t^T f(t,u)\,du}$$

Since we have only finitely many sources of noise, and infinitely many traded assets, there is a possibility of arbitrage in the bond market, unless the bond prices are inter–related in a specific way (which amounts to all bond prices having the same market price of risk, for all source of noise). This will impose conditions on the functions $\alpha$ and $\sigma$.

□

**Remarks 5.2** HJM is not a model, but a framework of models for the bond market; short rate models are another such framework. But whereas short rate models are generally Itô diffusions, and thus Markov processes, we can easily let $\alpha$ and $\sigma$ depend on past history. HJM models therefore need not be Markov models.(Of course, short rate models do not really need to be Markov either, but then their dynamics cannot be given by diffusions. We shall discuss the relationship between short rate and HJM models in the next section.)

□

For a market model (driven by Brownian motions) with only finitely many securities, we know that the model is arbitrage–free if and only if we can construct a risk–neutral measure, and complete if that measure is unique. Equivalently, the market is complete if an only if there

are as many traded risky securities as Brownian motions, subject to some conditions which ensure that the traded securities, are, in some sense , independent (where "independent" is meant in the sense of linear algebra, and not probability). The fact that there are only finitely many sources of noise, but infinitely many traded assets, means that the market is "over–complete", i.e. that there may be many ways of replicating a security. Unless all such replicating portfolios have the same price, there will be arbitrage. In practice, all securities must have the same market price of risk. If that's the case, we can construct a riskneutral measure (via a Girsanov transformation), which implies that the market is arbitrage–free. The arbitrage theory we've developed thus far only applies to markets with just finitely many traded securities, and it isn't at all clear that the impossibility of arbitrage implies the existence of a riskneutral measure (i.e. a measure under which all uncountably many zero coupon bond prices, when discounted, become martingales). We *can*, however, construct riskneutral measures for any finite subset of zero coupon bonds. Nevertheless, it is highly desirable to have a single riskneutral measure for all bonds simultaneously (because prices of securities are then just expected discounted payoffs, where the expectation is taken w.r.t. the riskneutral measure). We will therefore try to impose a strong form of the no–arbitrage condition: The existence of a riskneutral measure for all bonds.

To enable us to construct such a riskneutral measure, there must be relationships between $\alpha(t,T)$ and $\sigma(t,T)$ that must hold if the HJM model is to be arbitrage—free:

**Proposition 5.3** *Assume that the bond market is arbitrage–free in the strong sense, i.e. assume that there is a risk–neutral measure for bonds of all maturities. Then there is a (multidimensional) process $\lambda(t)$ such that, for all maturities $T$,*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)^{tr}\,ds + \sigma(t,T)\lambda(t)$$

**Proof:** Recall that

$$\frac{dp(t,T)}{p(t,T)} = \left[r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\right]\,dt + S(t,T)\,d\hat{W}_t$$

where $\hat{W}_t$ is a $\mathbb{P}$–Brownian motion. Here

$$A(t,T) = -\int_t^T \alpha(t,s)\,ds$$
$$S(t,T) = -\int_t^T \sigma(t,s)\,ds$$

If we use a Girsanov transformation with kernel $-\lambda$ to change to a new measure $\mathbb{Q}$, then new dynamics of $p(t,T)$ are

$$\frac{dp(t,T)}{p(t,T)} = \left[r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2 - S(t,T)\lambda(t)\right]\,dt + S(t,T)\,dW_t$$

where $W_t$ is a $\mathbb{Q}$–Brownian motion. For $\mathbb{Q}$ to be a riskneutral measure, each $p(t,T)$ must have drift $r(t)$, i.e.

$$A(t,T) + \frac{1}{2}||S(t,T)||^2 - \sigma(t,T)\lambda(t) = 0$$

This shows that $\lambda$ is just the market price of risk of $p(t,T)$ at time $t$, for all $T$: All bonds have the same market price of risk.

Differentiating this equation with respect to $T$ yields

$$-\alpha(t,T) + \sigma(t,T)\int_t^T \sigma(t,s)^{tr}\, ds + \sigma(t,T)\lambda(t) = 0$$

□

Suppose that we have an HJM model driven by $d$ sources of noise, so that each $\sigma(t,T)$ is a $d$–dimensional row vector $\sigma = (\sigma_1, \ldots, \sigma_d)$, and $\lambda = (\lambda_1, \ldots, \lambda_d)^{tr}$ is a $d$–dimensional column vector. We then have

$$\alpha(t,T) = \sum_{i=1}^{d} \sigma_i(t,T)\int_t^T \sigma_i(t,s)\, ds + \sum_{i=1}^{d} \sigma_i(t,T)\lambda_i(t) \tag{*}$$

If we take $\alpha$ and $\sigma$ as given, we can try and solve for $\lambda$. We then have uncountably many equations in just $d$ unknowns $\lambda_1(t), \ldots, \lambda_d(t)$ — one equation for each $T$. Thus $\alpha, \sigma$ cannot be specified arbitrarily. What we can do is

- Specify the volatility surface $\sigma(t,T)$.
- Choose $d$ benchmark maturities $T_1, \ldots, T_d$ and specify $\alpha(t,T_1), \ldots, \alpha(t,T_d)$.
- Solve the system $(*)$ of $d$ equations for the $d$ unknowns $\lambda_1(t), \ldots, \lambda_d(t)$.

All the other $\alpha(t,T)$ (for $T \neq$ a bench mark maturity) are now given by $(*)$.

## 5.2 Martingale Modelling

As for short rate models, it is often convenient to bypass the necessity of estimating the market price of risk, and to model directly under the risk–neutral measure $\mathbb{Q}$. i.e. we write

$$df(t,T) = \alpha(t,T)\, dt + \sigma(t,T)\, dW_t \qquad T \geq 0, 0 \leq t \leq T f(0,T) \quad = f^*(0,T)$$

where $W_t$ is a $\mathbb{Q}$–Brownian motion. Under $\mathbb{Q}$, the market price of risk is $\lambda = 0$, so we obtain:

**Proposition 5.4** (HJM Drift Conditions)
*The riskneutral dynamics of forward rates satisfy the following conditions:*

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)^{tr}\, ds$$

□

Thus in the riskneutral world, the drifts $\alpha(t,T)$ are completely determined by the volatility surface $\sigma(t,T)$. To create an HJM model, therefore, just follow the following steps:

- Estimate (or otherwise specify) a volatility surface $\sigma(t,T)$.
- Calculate the drifts $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)^{tr}\, ds$.

- Observe the term structure of forward rates $\{f^*(0,T) : T \geq 0\}$. This involves building a yield curve for all maturities.
- Integrate:
$$f(t,T) = f^*(0,T) + \int_0^t \alpha(u,T)\, du + \int_0^t \sigma(u,T)\, dW_u$$
- Compute bond prices $p(t,T) = e^{-\int_t^T f(t,s)\, ds}$ and the prices of other interest rate derivatives.

## 5.3 Examples and Applications

**Example 5.5** We consider here the simplest possible HJM model: We have only one source of noise, and put $\sigma(t,T) = \sigma =$ constant for all $t, T$. By the HJM drift conditions, we see that
$$\alpha(t,T) = \sigma \int_t^T \sigma\, ds = \sigma^2(T-t)$$
under the riskneutral measure. Hence the riskneutral dynamics of forward rates are
$$\begin{aligned} df(t,T) &= \sigma^2(T-t)\, dt + \sigma\, dW_t \\ f(0,T) &= f^*(0,T) \end{aligned}$$
Integrate this to obtain
$$\begin{aligned} f(t,T) &= f^*(0,T) + \sigma^2 t(T - \frac{t}{2}) + \sigma W_t \quad \text{so that} \\ r(t) &= f^*(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma W_t \end{aligned}$$
and thus the short rate dynamics are given by
$$dr_t = \left[f_T^*(0,t) + \sigma^2 t\right]\, dt + \sigma\, dW_t$$
These short rate dynamics should be familiar: We've obtained the Ho–Lee model fitted to the initial term structure! Note that we didn't have to do the actual fitting — in the HJM framework, fitting is automatic.

$\square$

Thus the Ho–Lee model is (equivalent to) the simplest HJM model.

**Example 5.6** Can the Hull–White (extended Vasiček) model be recast in the HJM framework?
Indeed it can. The Hull–White model $dr_t = (b(t) - ar_t)\, dt + \sigma\, dW_t$ is an affine term structure model, with bond prices $p(t,T) = e^{A(t,T)-B(t,T)r_t}$. Hence $f(t,T) = -A_T(t,T) + B_T(t,T)r_t$, which means
$$df(t,T) = [\cdot]\, dt + B_T(t,T)\sigma\, dW_t$$
where we haven't bothered to calculate the coefficient of the $dt$–term (which is, of course, just $\alpha(t,T)$). But for the Hull–White model, it was easy to calculate $B(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}]$, so that $B_T(t,T) = e^{-a(T-t)}$. It follows that
$$df(t,T) = \alpha(t,T)\, dt + \sigma e^{-a(T-t)}\, dW_t$$

Thus $\sigma(t,T) = \sigma e^{-a(T-t)}$ We can now use the HJM drift conditions to calculate $\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)\,ds = \frac{\sigma^2}{a}[e^{-a(T-t)} - e^{-2a(T-t)}]$.

To verify that the above model leads to the Hull–White model, recall that the short rate dynamics can be deduced from the forward rate dynamics as follows:

$$dr_t = [f_T(t,t) + \alpha(t,t)]\,dt + \sigma(t,t)\,dW_t$$

Now $\sigma(t,t) = \sigma$, and $\alpha(t,t) = 0$. Finally,

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)\,du + \int_0^t \sigma(u,T)\,dW_u$$

which implies that

$$r(t) = \Theta(t) + \int_0^t \sigma e^{-a(t-u)}\,dW_u$$

for some function $\Theta(t)$, and hence that

$$\begin{aligned} dr(t) &= \Theta'(t)\,dt - \left(a\int_0^t \sigma e^{-a(t-u)}\,dW_u\right)\,dt + \sigma dW_t \\ &= \left[\Theta'(t) - a(r(t) - \Theta(t))\right]\,dt + \sigma\,dW_t \\ &= [b(t) - ar(t)]\,dt + \sigma\,dW_t \end{aligned}$$

Moreover, $b(t) = \Theta'(t) + a\Theta(t)$, and $\Theta(t) = f(0,t) + \int_0^t \alpha(u,t)\,du = f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$. This is exactly the value of $b(t)$ which we obtained for the Hull–White model fitted to the initial term structure.

□

**Remarks 5.7** The above example suggests a simple mechanism for turning a fitted affine term structure model $dr_t = \mu_r\,dt + \sigma_r\,dW_t$ into an HJM model:

- If $p(t,T) = e^{A(t,T) - B(t,T)r_t}$, solve the (Riccatti) ODE for $B(t,T)$.
- Then the HJM volatility surface is given by $\sigma(t,T) = B_T(t,T)\sigma_r\,dW_t$.
- The HJM drift conditions now specify $\alpha(t,T)$ as well.

□

**Example 5.8** We consider a model with two sources of noise $W_t^1, W_t^2$ and a volatility surface

$$\sigma(t,T) = (\sigma_1, \sigma_2 e^{-a(T-t)})$$

where $\sigma_1, \sigma_2, a$ are positive constants. The HJM drift conditions dictate that

$$\alpha(t,T) = \sigma_1^2(T-t) + \frac{\sigma_2^2}{a}\left[e^{-a(T-t)} - e^{-2a(T-t)}\right]$$

Integrating the forward rate dynamics, we see

$$\begin{aligned} f(t,T) = f(0,T) + \sigma_1^2 t\left(T - \frac{t}{2}\right) + \frac{\sigma_2^2}{2a^2}\left[2e^{-aT}(1 - e^{at}) - e^{-2aT}(1 - e^{2at})\right] \\ + \sigma_1 W_t^1 + \sigma_2\int_0^t e^{-a(T-u)}\,dW_u^2 \end{aligned}$$

Thus

$$\begin{aligned}
r_t &= f(0,t) + \frac{\sigma_1^2 t^2}{2} + \frac{\sigma_2^2}{2a^2}\left[2(e^{-at}-1) - (e^{-2at}-1))\right] \\
&\quad + \sigma_1 W_t^1 + \sigma_2 \int_0^t e^{-a(t-u)}\, dW_u^2 \\
&= \Theta(t) + \sigma_1 W_t^1 + \sigma_2 \int_0^t e^{-a(t-u)}\, dW_u^2
\end{aligned}$$

Thus the short rate is a Gaussian process, and

$$\begin{aligned}
dr_t &= \left[\Theta'(t) - a\sigma_2 \int_0^t e^{-a(t-u)}\, dW_u^2\right] dt + \sigma_1\, dW_t^1 + \sigma_2\, dW_t^2 \\
&= \left[\Theta'(t) - a(r_t - \Theta(t) - \sigma_1 W_t^1)\right] dt + \sigma_1\, dW_t^1 + \sigma_2\, dW_t^2 \\
&= \left[b(t) - ar_t - a\sigma_1 W_t^1\right] dt + \sigma_1\, dW_t^1 + \sigma_2\, dW_t^2
\end{aligned}$$

This is not the form of one of our standard short rate models, because of the explicit presence of $W_t^1$ in the drift.

□

# 6 Market Models: Preliminaries

The HJM aproach studies the entire term structure of instantaneous forward rates $\{f(t,T) : t \leq T\}$, with considerable success, as we have seen. Nevertheless, forward rates for only a few maturities are available in the market, so the forward rate curve, like the instantaneous short rate, is a purely mathematical entity, a mathematical idealization. Market models, on the other hand, model observable (i.e. market–quoted) rates rather than idealized entities, and thus simple, discrete rates.

The London Interbank Offer Rates (LIBOR), for example, are quoted for different maturities (3–month, 6-month, etc.) and also for different currencies. These LIBOR spot rates imply LIBOR forward rates using an arbitrage argument. New LIBOR quotes are available daily. Swap rates (the fair rates for interest rate swaps) are another example of discrete market–quoted rates. The market model approach to interest rates dates back to Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997) and Jamshidian (also 1997). Several other approaches now exist, due to Hunt and Kennedy, and Musiela and Rutkowski, amongst others. It remains one of the most intensively researched areas of financial mathematics.

## 6.1 Black's Models

Black's model has long been the industry–standard model used by traders to price a variety of European–style options, including interest rate options, such as caps,floors, and swaptions. It is essentially a minor variation on the Black–Scholes formula, as we shall shortly see. Nevertheless, the suitability and adequacy of Black's model has often been questioned by academics, particularly in the arena of interest rate options.

Consider a European call option $C$ with strike $K$ and maturity $T$ on some market variable $X$. $X$ need not be a traded instrument — it could also be a market–quoted interest rate, for example. The main assumption is that $X_T$ is lognormally distributed *in the riskneutral*

*world.*. Thus we make no assumptions on the distribution of the process $(X_t)_t$ in general, but just on the value of $X$ at the expiry of the option. We further define the "volatility" of $X_T$ to be a non–negative number $\sigma$ satisfying

$$\text{variance of } \ln X_T = \sigma^2 T$$

Let $\mathbb{Q}$ be a riskneutral measure. Then the $t = 0$–value of the call is

$$C_0 = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_t\, dt}(X_T - K^+)\right]$$

Black uses two approximations to determine the value of $C_0$:

- Approximate

  $$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_t\, dt}(X_T - K)^+\right] \approx P(0,T)\mathbb{E}_{\mathbb{Q}}[(X_T - K)^+]$$

  i.e. discount outside the expectation operator.

- Now because $X_T$ is lognormal under $\mathbb{Q}$, we know that

  $$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[(X_T - K)^+] &= \mathbb{E}[X_T]N(d_1) - KN(d_2) \qquad \text{where} \\ d_1 &= \frac{\ln \frac{\mathbb{E}_{\mathbb{Q}}[X_T]}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

  Approximate

  $$\mathbb{E}_{\mathbb{Q}}[X_T] = \text{forward price/rate of } X = F_0$$

  i.e. approximate the expectation by the forward price/rate.
  Since the forward price of $X$ at time $T$ for time $T$ is just itself (i.e. $F_T = X_T$), this can be interpreted as saying that the forward rate process has zero drift, i.e. is a $\mathbb{Q}$–martingale.

Thus, using these two approximations, we obtain Black's model for a call on $X$:

$$\begin{aligned} C_0 &= P(0,T)[F_0 N(d_1) - KN(d_2)] \qquad \text{where} \\ d_1 &= \frac{\ln \frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

A similar formula is obtained for puts, using put–call parity.

If payments are based on a variable $X_T$, but only received at some later date $T^*$, then discounting must be done from time $T^*$ rather than from time $T$. Black's model then generalizes to give call prices

$$\begin{aligned} C_0 &= P(0,T^*)[F_0 N(d_1) - KN(d_2)] \qquad \text{where} \\ d_1 &= \frac{\ln \frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

where $F_0$ is still the $T$–forward value of $X$ at time $t = 0$. The appropriate generalized Black formula for put options follows once again by put–call parity.

Now it ought to be clear Black's model has several flaws. Firstly, it cannot be appropriate to use the first approximation when $X_T$ depends on interest rates, as it amounts to saying that

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_t\,dt}(X_T - K)^+\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r_t\,dt}\right]\mathbb{E}_{\mathbb{Q}}\left[(X_T - K)^+\right]$$

which is close to asserting that $r$ and $X_T$ are independent. That's a dangerous assumption if $X$ happens to be an interest rate derivative! There is no justification for the second approximation either. The expected value of $X_T$ under the riskneutral measure is its *futures price*, whereas the forward price is the expected value of $X_T$ under the $T$–forward riskneutral measure. These measures are not the same if interest rates are stochastic.

In spite of these flaws, Black's model remains heavily used — the industry standard. The method can be justified, provided that the relevant variable is taken to be lognormal under a different measure, associated with a different numéraire. We shall give several examples of this below. Review material on changes of measure and numéraire may be found in the next subsection.

**Example 6.1** Bond Options: Lognormal prices
We consider a call $C$ with strike $K$ and maturity $T$ on a coupon bearing bond $B$. We assume that the bond price at time $T$ is lognormally distributed (under the riskneutral measure), and that $\ln B_T$ has variance $\sigma^2 T$. This "volatility" $\sigma$ is obtained from historical data (or implied by other market variables).[1]

The $T$–forward bond price $F_0$ is simply the fair price which sets the value of a forward contract on $B$ equal to zero. A simple arbitrage argument shows

$$F_0 = \frac{B_0 - D}{P(0,T)}$$

where $B_0$ is the current value of the bond, $P(0,T)$ is the discount bond maturing at time $T$, and $D$ is the present value of all coupons (dividends) paid out during the life of the option. Thus Black's model determines

$$\begin{aligned}
C_0 &= (B_0 - D)N(d_1) - KP(0,T)N(d_2) \qquad \text{where} \\
d_1 &= \frac{\ln \frac{B_0 - D}{KP(0,T)} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}$$

The above call price is an *approximation* under the assumption that $B_T$ is lognormal under $\mathbb{Q}$, but *exact* if we assume lognormality of $B_T$ under the $T$–forward riskneutral measure $\mathbb{Q}^T$.

Of course, bond put options can be evaluated by put–call parity.

□

[1] In practice, yield volatilities are often obtained. If $\sigma_y$ is the volatility of the yield (i.e. if $\sigma_y^2 T$ is the standard deviation of the logarithm of the forward yield $\ln y_T$), then (with $D^* =$ duration) we have $\frac{\Delta B}{B_0} \approx -D^*\Delta y = -D^* y_0 \frac{\Delta y}{y_0}$, i.e. $\Delta(\ln B) \approx -D^* y_0 \Delta(\ln y)$. Thus the variance of $\ln B$ is approximately $(D^* y_0)^2 \times$ the variance of $\ln y$, i.e. $\sigma_B \approx D^* y_0 \sigma_y$.

**Example 6.2** Caps: Lognormal LIBOR Rates
An interest rate cap is an option–like contract which protects the holder against a floating interest rate moving too high. Each cap is a portfolio of caplets, each for a certain future time interval. A caplet is essentially a call option on the floating rate, given a certain cap rate as strike, based on a given notional amount. Consider, for example, a five–year cap, on a notional amount $A$, with cap rate $R$ and semiannual resets based on 6–month LIBOR. This is a portfolio of 10 caplets. The reset dates $T_0 = 0, T_1 = 0.5, T_2 = 1, \dots T_{10} = 5$ are referred to as the tenor structure of the cap. The $n^{\text{th}}$ caplet protects the holder against 6–month LIBOR rising above $R$ over the period $[T_{n-1}, T_n]$. It is a call option with strike $R$ on the 6–month spot LIBOR $L(T_{n-1})$ at time $T_{n-1}$, and will have the following payoff at time $T_n$:

$$\text{Payoff of } n^{\text{th}} \text{ cap} = A\delta_n(L(T_{n-1}) - R)^+ \qquad \text{where } \delta_n = T_n - T_{n-1}$$

(This is a *payment–in–arrears* cap. The first caplet is generally excluded from the cap, because there is no uncertainty about the spot LIBOR $L(T_0)$.)

To price the $n^{\text{th}}$ caplet using Black's model, we assume that the future spot LIBOR $L(T_{n-1})$ is lognormally distributed, with volatility $\sigma_{n-1}$. The $t = 0$–forward LIBOR rate (i.e. the $F_0$ of Black's model) for the period $[T_{n-1}, T_n]$ is given by

$$L(0, T_{n-1}) = \frac{P(0, T_{n-1}) - P(0, T_n)}{\delta_n P(0, T_n)}$$

(In this notation, the future spot rate, $L(T_{n-1})$, is just $L(T_{n-1}, T_{n-1})$.)
Hence the $t - 0$–value of the $n^{\text{th}}$ caplet is

$$\begin{aligned} C_n(0) &= A\delta_n P(0, T_n)\left[L(0, T_{n-1})N(d_{1,n-1}) - RN(d_{2,n-1})\right] \\ d_{1,n-1} &= \frac{\ln \frac{L(0,T_{n-1})}{R} + \frac{1}{2}\sigma_{n-1}^2 T_{n-1}}{\sigma\sqrt{T_{n-1}}} \\ d_{2,n-1} &= d_{1,n-1} - \sigma_{n-1}\sqrt{T_{n-1}} \end{aligned}$$

The price of the cap is therefore the *sum of the prices of the caplets* (though, as we have mentioned, the first cap is often excluded, i.e. $C_1(0)$ is set to zero).

The above price for a cap is an *approximation*, assuming that each future LIBOR spot rate $L(T_n)$ is lognormal under the riskneutral measure $\mathbb{Q}$. The formula for each caplet is *exact*, however, if it is assumed that $L(T_n)$ is lognormal under the $T_{n+1}$–forward measure. For then indeed

$$\frac{C_n(0)}{P(0, T_n)} = \mathbb{E}_{\mathbb{Q}^{T_n}}\left[\frac{A\delta_n[L(T_{n-1}) - R]^+]}{P(T_n, T_n)}\right]$$

which justifies the first approximation used in Black's model (i.e. discounting outside the expectation). Moreover, the second approximation is exact, i.e. the *forward* LIBOR rate $L(0, T_{n-1})$ is exactly equal to the expected value of the spot rate, but under the forward riskneutral measure: $L(0, T_{n-1}) = \mathbb{E}_{\mathbb{Q}^{T_n}}[L(T_{n-1})]$. To see this, note that a long forward rate agreement $F$, initiated at time $t = 0$ for period $[T_{n-1}, T_n]$, will have initial value $F_0 = 0$, and terminal value $F_{T_n} = \delta_n[L(T_{n-1}) - L(0, T_{n-1})]$. Hence

$$0 = \frac{F_0}{P(0, T_{n-1})} = \mathbb{E}_{\mathbb{Q}^{T_n}}\left[\frac{F_{T_n}}{P(T_n, T_n)}\right]$$

which yields the required result (because $L(0, T_{n-1})$ is a known constant).

So in order for the Black price of a cap to be accurate, we must simultaneously assume that each $L(T_n)$ is lognormal under $\mathbb{Q}^{T_{n+1}}$. This seems difficult to justify theoretically. One of the achievements of LIBOR market models is that they provide a framework under which these assumptions all *do* hold simultaneously, thus showing that the use of Black's model does not lead automatically to arbitrage opportunities.

□

**Example 6.3** Caps: Lognormal Bond Prices

A cap can be decomposed into a portfolio of puts on zero coupon bonds. To be precise, the $n^{\text{th}}$ caplet (from the previous example) has

$$\text{Payoff} = A\delta_n[L(T_{n-1}) - R]^+ \quad \text{at time } T_n$$

Since $L(T_{n-1})$ is known at time $T_{n-1}$ this is equivalent to a time–$T_{n-1}$ payoff of

$$\begin{aligned} \frac{A\delta_n[L(T_{n-1}) - R]^+}{1 + \delta_n L(T_{n-1})} &= A\left[1 - (1 + \delta_n R)P(T_{n-1}, T_n)\right]^+ \\ &= A(1 + \delta_n R)\left[\frac{1}{1 + \delta_n R} - P(T_{n-1}, T_n)\right]^+ \end{aligned}$$

This last line is easily seen to be the time–$T_{n-1}$ payoff of a portfolio of $A(1+\delta_n R)$–many put options with strike $\frac{1}{1+\delta_n R}$ and expiry $T_{n-1}$ on underlying security $P(t, T_n)$. If at time $T_{n-1}$ the caplet has the same payoff as a portfolio of puts on $P(t, T_n)$, then, by the Law of One Price, the value of the caplet must have the same value as the portfolio of puts at any earlier time as well.

Thus the $t = 0$–value of the $n^{\text{th}}$ caplet is

$$\begin{aligned} C_n(0) = A(1 + \delta_n R) \times &\text{ value of put option on } P(t, T_n) \text{ with strike } \frac{1}{1 + \delta_n R} \\ &\text{and expiry } T_{n-1} \end{aligned}$$

This can be evaluated using the method of the first example of this subsection.

□

**Example 6.4** Swaptions: Lognormal Swap Rates

Suppose we initiate, at time $t$, a pay–fixed interest rate swap starting at time $T \geq t$, with tenor structure $T = T_0 < T_1 < \cdots < T_N$ on a notional amount $A$. This is known as a *forward swap* or *deferred swap*. Let $\delta_n = T_n - T_{n-1}$, and recall that at $T_n$ pay–fixed receives

$$A\delta_n(L(T_{n-1}) - S_{t,T}) \qquad n = 1, \ldots, N$$

where $S_{t,T}$ is the $T$–forward swap rate at time $t$, and $L(T_{n-1})$ is the spot LIBOR rate at time $T_{n-1}$ for the period $[T_{n-1}, T_n]$. Further recall that $S_{t,T}$ is the rate which sets the initial (i.e. time $t$) value of the forward swap equal to zero.

The interest payments on a pay–fixed swap are equivalent to the payments of a portfolio consisting of short a coupon bond with coupon rate $S_{t,T}$, and long a floating rate note. The

bond and the FRN both come into existence at time $T$. The current value of such a forward starting bond bond is

$$A[\sum_{n=1}^{N}\delta_n S_{t,T}P(t,T_n)+P(t,T_N)]$$

The floating rate note will trade at par at time $T$, i.e. we need to set aside $AP(t,T)$ at time $t$ to purchase the FRN at time $T$. Hence the forward swap rate satisfies

$$-A[\sum_{n=1}^{N}\delta_n P(t,T_n)S_{t,T}+P(t,T_N)]+AP(t,T)=0$$

(where the coupon bond and FRN have the same payment dates as the swap, and the same notional) and thus

$$S_{t,T}=\frac{P(t,T)-P(t,T_N)}{\sum_{n=1}^{N}\delta_n P(t,T_n)}$$

If $t=T$, then $S_{t,t}$ is just the ordinary spot swap rate at time $t$.

A swaption $C$ is the right to enter into a pay–fixed swap at some future date $T$ at a strike rate $R$. If the tenor structure is $T=T_0<T_1<T_2<\cdots<T_N$, then the swaption gives the holder the right (but not the obligation) to receive at each of the dates $T_1,\ldots,T_N$ an amount

$$A\delta_n(L(T_{n-1})-R)$$

If a pay–fixed swap were to be entered at time $T$ at the spot swap rate, then payments would be

$$A\delta_n(L(T_{n-1})-S_{T,T})$$

and thus the swaption would be exercised only if $R<S_{T,T}$. The swaption thus gives rise to a series of payments

$$A\delta_n(S_{T,T}-R)^+$$

at times $T_n$. Each payment is equivalent to the payoff of $A\delta_n$–many calls with strike $R$ and maturity $T$ on underlying $S_{T,T}$. Using the generalized version of Black's model, i.e. assuming that $S_{T,T}$ is lognormal under the riskneutral measure and making the appropriate approximations, the $t=0$–value of each such payment is

$$A\delta_n P(0,T_n)\,[S_{0,T}N(d_1)-RN(d_2)]$$

where $d_1=\frac{\ln\frac{S_{0,T}}{R}+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}$, $d_2=d_1-\sigma\sqrt{T}$, and $\sigma$ is the volatility of the future spot swap rate $S_{T,T}$. Hence the value of the swaption is

$$\begin{aligned}
C_0 &= \sum_{n=1}^{N}A\delta_n P(0,T_n)[S_{0,T}N(d_1)-RN(d_2)] \qquad \text{where}\\
d_1 &= \frac{\ln\frac{S_{0,T}}{R}+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}\\
d_2 &= d_1-\sigma\sqrt{T}
\end{aligned}$$

and

$$S_{0,T}=\frac{P(t,T)-P(t,T_N)}{\sum_{n=1}^{N}\delta_n P(0,T_n)}$$

We saw that we can make the Black formula for caps exact, provided we work with the appropriate numéraires, under the appropriate equivalent martingale measures. Can we make Black's formula for swaptions exact? Yes, indeed. Note that the numerator in the expression for $S_{0,T}$ is equivalent to a portfolio of zero coupon bonds, i.e.

$$\sum_{n=1}^{N} \delta_n P(0, T_n)$$

corresponds to a stream of cashflows of size $\delta_n$ at time $T_n$. If, as is often the case, all the $\delta_n$ are of the same size, then this portfolio is just an annuity. Now we may think of the portfolio as a traded asset, call it $X$, and use it as numéraire.

The first of the Black approximations is exact under the measure $\mathbb{Q}_X$: The time–$T$ value of all the payoffs of the swaption is

$$C_T = \sum_{n=1}^{N} A\delta_n P(T, T_n)[S_{T,T} - R]^+ = AX_T[S_{T,T} - R]^+$$

Hence

$$\frac{C_0}{X_0} = \mathbb{E}_{\mathbb{Q}_X}\left[\frac{C_T}{X_T}\right] = \mathbb{E}_{\mathbb{Q}_X}[A(S_{T,T} - R)^+]$$

so that

$$C_0 = \sum_{n=1}^{N} A\delta_n P(0, T_n)\mathbb{E}_{\mathbb{Q}_X}[(S_{T,T} - R)^+]$$

i.e. we discount outside the expectation.
As for the second approximation, we need to show that the forward swap rate $S_{0,T}$ (which can now be seen to equal $\frac{P(0,T)-P(0,T_N)}{X_0}$) is just the expected value of the future spot swap rate $S_{T,T}$ under the EMM $\mathbb{Q}_X$, i.e. that $\mathbb{E}_{\mathbb{Q}_X}[S_{T,T}] = S_{0,T}$. To see this, consider a pay–fixed forward swap $F$ initiated at $t = 0$ to start at time $T$, with interest payment dates $T_1, \ldots, T_N$. The $t = 0$–value of the contract is $F_0 = 0$, whereas at time $T$ the value is $F_T = \sum_{n=1}^{N} A\delta_n P(T, T_n)[S_{T,T} - S_{0,T}] = AX_T[S_{T,T} - S_{0,T}]$. The desired result now follows immediately from the fact that

$$0 = \frac{F_0}{X_0} = \mathbb{E}_{\mathbb{Q}_X}\left[\frac{F_T}{X_T}\right]$$

Hence Black's formula is exact, provided we assume that swap rates are lognormally distributed under the EMM associated with the annuity process $X_t = \sum_{n=1}^{N} \delta_n P(t, T_n)$.

□

It's pretty amazing that the Black formula for various derivatives (published in 1976) can in many cases be made exact using the change of numéraire technique (discovered in the early 1990's). In particular, both the Black formula for caps and that for swaptions are exact if we assume that LIBOR rates are lognormal under the appropriate forward riskneutral measures, and that swap rates are lognormal under the "annuity" measure.

## 6.2 Review of Changes of Measure and Numéraire; LIBOR Rates

Fix a horizon $T^* > 0$ and suppose that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t, (S_t^i)_{i,t})$ is a market model, where the filtration $(\mathcal{F}_t)_t$ is generated by a standard (multi–dimensional) $\mathbb{P}$–Brownian motion $(W_t)_t$, augmented to satisfy the usual conditions. Let $\mathbb{Q}$ be the riskneutral measure, i.e. a measure which has the property that all asset price processes $S_t^i$ are martingales when denominated in units of the money market account $A_t$. We briefly recall some facts about how Girsanov's Theorem is used to change the measure (e.g. to construct $\mathbb{Q}$ from $\mathbb{P}$):

- Assume that the asset dynamics are given by

$$\frac{dS_t^i}{S_t^i} = \mu^i(t, S_t)\ dt + \sigma^i(t, S_t)\ dW_t \qquad \frac{dA_t}{A_t} = r_t A_t\ dt$$

  with suitable initial conditions.
  Recall that the market price of risk $\lambda_t^{\mathbb{P}}$ is a vector satisfying

$$\sigma_t^i \cdot \lambda_t^{\mathbb{P}} = \mu_t^i - r_t$$

  (This looks like it depends on the asset $S^i$, but we know from previously developed theory that, for a model to be arbitrage–free, all assets must have the same market price of risk. Hence we've suppressed an index $i$.)

- Let $u(t, \omega)$ be a predictable process, to be used as a kernel for a Girsanov transformation.

- Define a new measure $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}_{T^*}\left(\int_0^{\cdot} u_t\ dW_t\right) = e^{\int_0^T u_t\ dW_t - \frac{1}{2}\int_0^T ||u_t||^2\ dt}$$

- Girsanov's Theorem states that

$$\tilde{W}_t = W_t - \int_0^t u_s\ ds$$

  is a $\tilde{\mathbb{P}}$–Brownian motion.

- Thus the new asset dynamics are, under $\tilde{\mathbb{P}}$,given by

$$\frac{dS_t^i}{S_t^i} = (\mu_t^i + \sigma_t^i u_t)\ dt + \sigma_t^i\ d\tilde{W}_t \qquad \frac{dA_t}{A_t} = r_t A_t\ dt$$

  It follows that the market price of risk under $\tilde{\mathbb{P}}$ must satisfy the relation

$$\lambda_t^{\tilde{\mathbb{P}}} \sigma_t^i = \mu_t^i + \sigma_t^i u_t - r_t = (\lambda_t^{\mathbb{P}} + u_t)\sigma_t^i$$

  and thus

$$\lambda_t^{\tilde{\mathbb{P}}} = \lambda_t^{\mathbb{P}} + u_t$$

- Hence a Girsanov transformation adds the Girsanov kernel to the market price of risk. It adds volatility $\times$ kernel to the drift.

- To obtain a riskneutral measure $\mathbb{Q}$, the new market price of risk $\lambda_t^{\mathbb{Q}}$ must be zero, and thus we must have $u_t = -\lambda_t^{\mathbb{P}}$. This is in agreement with what we found earlier. In that case, the drift becomes $\mu^i - \sigma_t^i \lambda_t^{\mathbb{P}} = r_t$, which we already know very well.

- To change from the riskneutral measure $\mathbb{Q}$ to an equivalent martingale measure $\mathbb{Q}_X$ for numéraire $X$, we proceed as follows: Start in the riskneutral world, where $\frac{dS^t}{S_t} = r\ dt + \sigma_S\ dW_t^{\mathbb{Q}}$, and $\frac{dX_t}{X_t} = r\ dt + \sigma_X\ dW_t^{\mathbb{Q}}$. Under $\mathbb{Q}_X$, the ratios $\hat{S}_t = \frac{S_t}{X_t}$ are martingales. Now under $\mathbb{Q}$, the ratios have dynamics

$$\begin{aligned}\frac{d\hat{S}_t}{\hat{S}_t} &= -\sigma_X(\sigma_S - \sigma_X)\ dt + (\sigma_S - \sigma_X)\ dW_t^{\mathbb{Q}} \\ &= -\sigma_X \hat{\sigma}\ dt + \hat{\sigma}\ dW_t^{\mathbb{Q}}\end{aligned}$$

  (where $\hat{\sigma} = \sigma_S - \sigma_X$). To make the drift equal to zero (i.e. to make $\hat{S}_t$ into a martingale), we need to to add $\hat{\sigma} \times \sigma_X =$ volatility $\times \sigma_X$, i.e. we need to use a Girsanov transformation with kernel $\sigma_X$. Thus

$$\frac{d\mathbb{Q}_X}{d\mathbb{Q}} = \mathcal{E}_T(\int_0^{\cdot} \sigma_X\ dW_t^{\mathbb{Q}})$$

- Hence we need to add $\sigma_X$ to the riskneutral market price of risk to obtain the market price of risk under $\mathbb{Q}_X$. Since the riskneutral market price of risk is zero, the market price of risk under $\mathbb{Q}_X$ is just the volatility of the numéraire $X$.

- Numéraire–denominated asset price dynamics under the associated equivalent martingale measure are therefore just

$$\frac{d\hat{S}_t}{S_t} = (\sigma_S - \sigma_X)\ dW_t^{\mathbb{Q}_X}$$

- If the numéraire is the $T$–bond $P(t,T)$, the associated EMM is called the $T$–forward riskneutral measure, and denoted by $\mathbb{Q}^T$. If bond price dynamics are

$$\frac{dP(t,S)}{P(t,S)} = \mu_S(t)\ dt + \sigma_S(t)\ dW_t^{\mathbb{P}}$$

  under the "real–world" measure $\mathbb{P}$, then the numéraire denominated dynamics are given by

$$\frac{d\hat{P}(t,S)}{\hat{P}(t,S)} = (\sigma_S - \sigma_T)\ dW_t^T$$

  where $\hat{P}(t,S) = \frac{P(t,S)}{P(t,T)}$ and $W_t^T$ is a $\mathbb{Q}^T$–Brownian motion.

- Given future times $T < S$, the market price of risk under $\mathbb{Q}^S$ is just $\sigma_S$, whereas the market price of risk under $\mathbb{Q}^T$ is $\sigma_T$. To move from $\mathbb{Q}^S$–world to $\mathbb{Q}^T$–world, we must change the market price of risk from $\sigma_S$ to $\sigma_T$, i.e. we need to add $\sigma_T - \sigma_S$ to the market price of risk under $\mathbb{Q}^S$. Hence the change from $\mathbb{Q}^S$–world to $\mathbb{Q}^T$–world is effected by a Girsanov transformation with kernel $\sigma_T - \sigma_S$, i.e.

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}^S} = \mathcal{E}_T\left(\int_0^{\cdot} \sigma_T - \sigma_S\ dW_t^{\mathbb{Q}^S}\right)$$

We can also verify this directly. Recall that the Radon–Nikodym process $\xi_t = \mathbb{E}_{\mathbb{Q}^S}[\frac{d\mathbb{Q}^T}{d\mathbb{Q}^S}|\mathcal{F}_t]$ for a change of numéraire is given by a ratio of asset ratios:

$$\xi_t = \frac{P(t,T)/P(t,S)}{P(0,T)/P(0,S)}$$

and thus

$$d\xi_t = \xi_t[\sigma_T(t) - \sigma_S(t)]\, dW_t^{\mathbb{Q}^S}$$

The solution of this SDE, together with the initial condition $\xi_0 = 1$, is just $\xi_t = \mathcal{E}_t\left(\int_0^\cdot \sigma_T(u) - \sigma_S(u)\, dW_u^{\mathbb{Q}^S}\right)$.

- Finally, note that the asset ratio process $\check{P}(t,T) = \frac{P(t,T)}{P(t,S)}$ satisfies the same SDE as does $\xi_t$

$$\frac{d\check{P}(t,T)}{\check{P}(t,T)} = [\sigma_T(t) - \sigma_S(t)]\, dW_t^{\mathbb{Q}^S}$$

although their initial conditions may differ. Hence $\xi_t$ and $\check{P}(t,T)$ differ by a constant factor, i.e.

$$\mathbb{E}_{\mathbb{Q}^S}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}^S}|\mathcal{F}_t\right] = \xi_t = c\check{P}(t,T) = \frac{cP(t,T)}{P(t,S)}$$

Let $T^* > 0$ be a a horizon for our bond market model. The time–$t$ forward LIBOR rate for the future interval $[T, T+\delta]$ (where $T \le T^* - \delta$) is defined by

$$1 + \delta L(t,T) = \frac{P(t,T)}{P(t,T+\delta)} \quad \text{i.e.} \quad L(t,T) = \frac{P(t,T) - P(t,T+\delta)}{\delta P(t,T+\delta)}$$

We saw earlier that $L(t,T)$ is the interest rate for the period $[T, T+\delta]$ that can be locked in at time $t$ (by a judicious investment in a portfolio of $T$– and $T+\delta$–bonds with zero initial cost).

Alternatively, the forward LIBOR rate $L(t,T)$ can be regarded as the swap rate for a *single–period swap settled in arrears*. For suppose that we have a single–period interest rate swap, contracted at time $t$, for the period $[T, T+\delta]$, to be settled at time $T+\delta$. Thus, at time $T+\delta$, the pay–fixed side pays $\delta R$, and the receive–fixed party pays $P^{-1}(T, T+\delta) - 1$, where $R$ is the fair swap rate, and $P^{-1}(T, T+\delta) = 1 + \delta S$, $S = L(T,T)$ the spot rate at time $T$ for period $T, T+\delta]$. Equivalently, by adding 1 to both payments, pay–fixed pays $Y^{fx}$ and receive–fixed pays $Y^{fl}$, where

$$Y^{fx} = 1 + \delta R \qquad Y^{fl} = P^{-1}(T, T+\delta)$$

We can regard $Y^{fx}$ and $Y^{fl}$ as contingent claims which are paid out at time $T+\delta$. It is clear that the time $t$–value of $Y^{fx}$ is just

$$Y^{fx} = P(t, T+\delta)[1 + \delta R]$$

The time–$t$ value of $Y^{fl}$ is obtained as follows: If, at time $T$, we invest \$1.00 in $T+\delta$–bonds, the payoff at time $T+\delta$ will be $P^{-1}(T, T+\delta)$. To obtain the required \$1.00, we must invest in one $T$–bond at time $t \le T$. Hence

$$Y_t^{fl} = P(t,T)$$

The swap rate at time $t$ is the rate $R$ for which $Y^{fx} = Y^{fl}$, and thus $R = \frac{P(t,T)-P(t,T+\delta)}{\delta P(t,T+\delta)} = L(t,T)$.

Define

$$P(t,T,S) = \frac{P(t,T)}{P(t,S)} = 1 + \delta L(t,T) \qquad \text{for } t \le T \le S \text{ and } \delta = S - T$$

Then $P(t,T,S)$ is a $\mathbb{Q}^S$–martingale. In particular, the LIBOR forward rate $L(t,T)$ is a $\mathbb{Q}^{T+\delta}$–martingale. Thus the LIBOR forward rate $L(t,T)$ is simply the expected value of the LIBOR spot rate $L(T,T)$ at time $T$, where the expectation is taken under the $\mathbb{Q}^{T+\delta}$–measure.

# 7 Lognormal Forward LIBOR Market Models

We start with a pre–specified sequence of times

$$0 = T_0 < T_1 < T_2 < \cdots < T_N = T^*$$

These times, typically settlement– or reset dates, are collectively known as the *tenor structure*. We also define $\delta_j = T_j - T_{j-1}$ for $j = 1, \ldots, N$. Then the forward LIBOR rate satisfy

$$1 + \delta_{j+1} L(t,T_j) = \frac{P(t,T_j)}{P(t,T_{j+1})} = P(t,T_j,T_{j+1})$$

We assume that the bond market satisfies a strong form of the no–arbitrage condition, i.e. we assume that there exists a riskneutral measure $\mathbb{Q}$ simultaneously for all discount bonds $P(t,T)$. We denote, for each $P(t,T)$, its associated forward riskneutral measure by $\mathbb{Q}^T$. $W_t$ and $W_t^T$ will denote, respectively, $\mathbb{Q}$– and $\mathbb{Q}^T$–Brownian motions.

Let $S(t,T)$ be the volatility of the $T$–bond $P(t,T)$ at time $t$. From the previous subsection, we know the following:

- $\mathbb{Q}^{T_j}$ is obtained from $\mathbb{Q}^{T_{j+1}}$ via a Girsanov transformation with kernel $S(t,T_j) - S(t,T_{j+1})$, i.e.
$$\frac{d\mathbb{P}^{T_j}}{d\mathbb{P}^{T_{j+1}}} = \mathcal{E}_{T_j}\left(\int_0^{\cdot} S(t,T_j) - S(t,T_{j+1})\, dW_t^{T_{j+1}}\right)$$
- Each asset ratio $P(t,T_j,T_{j+1}) = \frac{P(t,T_j)}{P(t,T_{j+1})}$ is a $\mathbb{Q}^{T_{j+1}}$–martingale.
- Each forward LIBOR rate $L(t,T_j)$ is a $\mathbb{Q}^{T_{j+1}}$–martingale.
- The $\mathbb{Q}^{T_{j+1}}$–dynamics of the asset ratio $P(t,T_j,T_{j+1})$ are
$$\frac{dP(t,T_j,T_{j+1})}{P(t,T_j,T_{j+1})} = (S(t,T_j) - S(t,T_{j+1}))\, dW_t^{T_{j+1}}$$
- There is a constant $c$ such that the Radon–Nikodym process and the asset ratio process are related
$$\mathbb{E}_{\mathbb{Q}^{T_{j+1}}}\left[\frac{d\mathbb{Q}^{T_j}}{d\mathbb{Q}^{T_{j+1}}}|\mathcal{F}_t\right] = cP(t,T_j,T_{j+1}) = c(1 + \delta_{j+1} L(t,T_j))$$

Note that, assuming that the forward LIBOR rate processes $L(t,T)$ are strictly positive, we have the following dynamics:

$$dL(t,T_j) = L(t,T_j)\lambda(t,T_j)\, dW_t^{T_{j+1}}$$

This follows from the Martingale Representation Theorem: $L(t,T_j)$ is a $\mathbb{Q}^{T_{j+1}}$–martingale, and thus we must have $dL(t,T_j) = h_t\, dW_t^{T_{j+1}}$. Since $L(t,T_j)$ is strictly positive, we may define $\lambda(t,T_j) = frach_t L(t,T_j)$ to obtain $dL(t,T_j) = L(t,T_j)\lambda(t,T_j)\, dW_t^{T_{j+1}}$.

Now $P(t,T_j,T_{j+1}) = 1+\delta_{j+1}L(t,T_j)$, so that $dP(t,T_j,T_{j+1}) = \delta_{j+1}\, dL(t,T_j) = \delta_{j+1}L(t,T_j)\lambda(t,T_j)\, dW_t^{T_{j+1}}$. We also found that $\frac{dP(t,T_j,T_{j+1})}{P(t,T_j,T_{j+1})} = (S(t,T_j) - S(t,T_{j+1}))\, dW_t^{T_{j+1}}$, and equating these expressions yields

$$\frac{\delta_{j+1}L(t,T_j)}{1+\delta_{j+1}L(t,T_j)}\lambda(t,T_j) = S(t,T_j) - S(t,T_{j+1})$$

This expression will play an important role in the inductive construction of lognormal models of forward LIBOR rates.

Since the move from $\mathbb{Q}^{T_{j+1}}$–world to $\mathbb{Q}^{T_j}$–world is accomplished by a Girsanov transformation with kernel $S(t,T_j) - S(t,T_{j+1}) = \frac{\delta_{j+1}L(t,T_j)\lambda(t,T_j)}{1+\delta_{j+1}L(t,T_j)}$, the dynamics of $L(t,T_j)$ under $\mathbb{Q}^{T_j}$ are given by

$$dL(t,T_j) = L(t,T_j)\left[\frac{\delta_{j+1}||\lambda(t,T_j)||^2}{1+\delta_{j+1}L(t,T_j)}\, dt + \lambda(t,T_j)\, dW_t^{T_j}\right]$$

because volatility $\times$ kernel must be added to the $\mathbb{Q}^{T_{j+1}}$–drift of $L(t,T_j)$, while leaving the volatility unchanged (and the drift is zero, while the volatility is $\lambda(t,T_j)$).

## 7.1 The Brace–Gatarek–Musiela Approach to Forward LIBOR

> *In most markets, caps and floors form the largest component of an average swap derivatives book.... Market practice is to price the option assuming that the underlying forward rate process is lognormally distributed with zero drift. Consequently, the option price is given by the Black futures formula, discounted from the settlement data.*
> *In an arbitrage–free setting, forward rates over consecutive intervals are all related to one another, and cannot all be lognormal under one arbitrage–free measure. That is probably what led the academic community to a degree of skepticism toward the market practice of pricing caps...*
> *The aim of this paper is to show that market practice can be made consistent with an arbitrage–free term structure model... This is possible because each rate is lognormal under the forward (to the settlement date) arbitrage–free measure rather than under one (spot) arbitrage–free measure. Lognormality under the appropriate forward and not spot arbitrage–free measure is needed to justify the Black futures formula with discount for caplet pricing.*
>
> — Brace, Gatarek, Musiela [1997]

The BGM–model starts from a family $P(t,T)$ of discount bond prices up to some horizon maturity $T^*$. We assume that each forward rate is over a period of length $\delta$ (the same for all

rates). The bond price processes also give us the bond ratio processes (i.e. forward prices) $P(t,T,S) = \frac{P(t,T)}{P(t,S)}$. The forward LIBOR rates $L(t,T)$ are thus defined by

$$1 + \delta L(t,T) = P(t,T,T+\delta) \qquad \text{for } T \leq T^* - \delta$$

BGM put their model inside the HJM framework, i.e. they assume that a term structure of instantaneous forward rates for all maturities (less than the horizon date $T^*$) is available. In contrast, the Musiela–Rutkowski and Jamshidian approaches require forward rates only for a discrete set of tenor dates, as we shall see. Now recall that if, in an HJM model, the riskneutral dynamics of the instantaneous forward rate $f(t,T)$ is given by

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t \quad \text{where } \alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)\,du$$

(using the HJM drift condition), then the riskneutral bond price dynamics are given by

$$\frac{dP(t,T)}{P(t,T)} = r_t\,dt + S(t,T)\,dW_t \quad \text{where } S(t,T) = -\int_t^T \sigma(t,u)\,du$$

Further recall that earlier we obtained

$$\frac{\delta L(t,T)}{1+\delta L(t,T)}\lambda(t,T) = S(t,T) - S(t,T+\delta)$$

(which also follows if we apply Itô's formula to the identity $1 + \delta L(t,T) = e^{\int_T^{T+\delta} f(t,u)\,du}$ and compare the $dW_t$–terms). The main problem is this:

> How can we specify bond volatilities $S(t,T)$ (or equivalently, the instantaneous forward rate volatilities $\sigma(t,T) = -\frac{\partial S(t,T)}{\partial T}$) so that the resulting *discrete simple* forward LIBOR rates will have the desired deterministic volatity structure?

We have already seen that $L(t,T)$ is a non–negative $\mathbb{Q}^{T+\delta}$–martingale. For $1+\delta L(t,T) = P(t,T,T+\delta)$, and so $dL(t,T) = \delta^{-1}dP(t,T,T+\delta)$. But $P(t,T,T+\delta)$ is a $\mathbb{Q}^{T+\delta}$–martingale (by definition of $\mathbb{Q}^{T+\delta}$), with dynamics $\frac{dP(t,T,T+\delta)}{P(t,T,T+\delta)} = [S(t,T) - S(t,T+\delta)]\,dW_t^{T+\delta}$. It therefore follows that

$$\begin{aligned} dL(t,T) &= \delta^{-1}P(t,T,T+\delta)[S(t,T) - S(t,T+\delta)]\,dW_t^{T+\delta} \\ &= L(t,T)\left(\frac{[1+\delta L(t,T)][S(t,T) - S(t,T+\delta)]}{\delta L(t,T)}\right)\,dW_t^{T+\delta} \end{aligned}$$

i.e.

$$\begin{aligned} dL(t,T) &= L(t,T)\lambda(t,T)\,dW_t^{T+\delta} \\ \text{where } \lambda(t,T) &= \frac{[1+\delta L(t,T)][S(t,T) - S(t,T+\delta)]}{\delta L(t,T)} \end{aligned}$$

We are therefore able to derive the forward LIBOR dynamics directly from the bond price volatilities (or, equivalently, the instantaneous forward rate volatilities). Since the forward

riskneutral measure $\mathbb{Q}^{T+\delta}$ is obtained from the (spot) riskneutral measure $\mathbb{Q}$ by a Girsanov transformation with kernel $S(t, T+\delta)$, we have

$$dW_t^{T+\delta} = dW_t - S(t, T+\delta)\ dt$$

for a $\mathbb{Q}$–Brownian motion $W_t$. Thus the riskneutral drift is directly determined by the volatility structure (as it is in the HJM model), giving riskneutral forward LIBOR rate dynamics

$$\frac{dL(t,T)}{L(t,T)} = -\lambda(t,T) \cdot S(t, T+\delta)\ dt + \lambda(t,T)\ dW_t$$

Now suppose that we want to create an HJM model in which forward LIBOR rates $L(t,T)$ have a deterministic volatility structure $\lambda(t,T)$. Above, we found that

$$S(t,T) - S(t,T+\delta) = \int_T^{T+\delta} \sigma(t,u)\ du = \frac{\delta L(t,T)}{1+\delta L(t,T)} \lambda(t,T)$$

(where $S$ and $\sigma$ are the bond and instantaneous forward rate volatilities respectively). In order to find the bond volatilities, it is necessary to impose some additional conditions. Set

$$\sigma(t,u) = 0 \quad \text{when } 0 \leq u - t \leq \delta$$

(This is the fundamental assumption made in BGM(1997)).
Now find the bond volatilities by a recursive procedure:

- Choose $n$ such that $n\delta \leq T - t < (n+1)\delta$. Equivalently $n = \sup\{k \in \mathbb{N} : k\delta \leq T - t\} = [\delta^{-1}(T-t)]$ (where $[x]$ is the integer part of $x$).

- Then $S(t, T-n\delta) = -\int_t^{T-n\delta} \sigma(t,u)\ du = 0$, because $0 \leq u - t \leq \delta$ when $t \leq u \leq T - n\delta$.

- Thus

$$\begin{aligned} S(t,T) = [S(t,T) - S(t,T-\delta)] + [S(t,T-\delta) - S(t,T-2\delta)] + \dots \\ \dots + [S(t,T-(n-1)\delta) - S(t,T-n\delta)] \end{aligned}$$

  implies

$$\begin{aligned} S(t,T) = -\frac{\delta L(t,T-\delta)}{1+\delta L(t,T-\delta)}\lambda(t,T-\delta) - \frac{\delta L(t,T-2\delta)}{1+\delta L(t,T-2\delta)}\lambda(t,T-2\delta) - \dots \\ \dots - \frac{\delta L(t,T-n\delta)}{1+\delta L(t,T-n\delta)}\lambda(t,T-n\delta) \end{aligned}$$

- i.e.

$$S(t,T) = -\sum_{k=1}^{[\delta^{-1}(T-t)]} \frac{\delta L(t,T-k\delta)}{1+\delta L(t,T-k\delta)}\lambda(t,T-k\delta)$$

Equivalently,

(i) Define $S(t,T) = 0$ for $0 \leq T - t < \delta$.

(ii) Then define $S(t,T) = S(t,T-\delta) - \frac{\delta L(t,T-\delta)}{1+\delta L(t,T-\delta)}\lambda(t,T-\delta)$ for $\delta \leq T-t < 2\delta$.
(Note that if $\delta \leq T-t < 2\delta$, then $0 \leq (T-\delta)-t < \delta$, so $S(t,T-\delta)$ has already been defined.)

(iii) Then define $S(t,T) = S(t,T-\delta) - \frac{\delta L(t,T-\delta)}{1+\delta L(t,T-\delta)}\lambda(t,T-\delta)$ for $2\delta \leq T-t < 3\delta$.
(Note that if $2\delta \leq T-t < 3\delta$, then $\delta \leq (T-\delta)-t < 2\delta$, so $S(t,T-\delta)$ has already been defined.)

(iv) ... etc.

In this way, if we specify bond volatilities by this *forward induction*, then we will have an HJM model in which the forward LIBOR rates $L(t,T)$ have the required deterministic volatilities $\lambda(t,T)$. Since each $L(t,T)$ is a strictly postive $\mathbb{Q}^{T+\delta}$–martingale, it follows that each $L(t,T)$ is lognormal under $\mathbb{Q}^{T+\delta}$, and thus that the Black formula for caps is valid in this model.

## 7.2 The Musiela–Rutkowski Approach to Forward LIBOR

Unlike the BGM–approach, which lies within the HJM framework and specifies a model of forward LIBOR rates $L(t,T)$ for all maturities $T$ (below the horizon $T^*$), the Musiela Rutkowski (MR) approach only specifies LIBOR rates for a discrete set of maturities. We start with a discrete tenor structure

$$0 < T_0 < T_1 < \cdots < T_N = T^* \qquad \delta_n = T_n - T_{n-1}$$

and define $T_{-1} = 0$ (for ease of handling certain formulas). We further assume that we are *given*

- A family of bounded adapted processes $\lambda(t,T_n)$ for $n = 0,\dots,N-1$ which represent the volatilities of the forward LIBOR rates $L(t,T_n)$.
- An initial term structure $P(0,T_n)$ of discount bond prices (used to specify the initial conditions of the SDE's which we will write down for the LIBOR rates). We further assume that $P(0,T_0) > P(0,T_1) \cdots > P(0,T_N)$.

In contrast to the BGM approach, we do not need a bond price dynamics at all, i.e. we will attempt to model LIBOR rates directly.

Before we construct the MR model of LIBOR rates, a lemma which will prove useful

**Lemma 7.1** *If $X, Y$ are adapted processes*

$$dX_t = \alpha_t\, dW_t \qquad dY_t = \beta_t\, dW_t$$

*and if $Z_t = \frac{1}{1+Y_t}$, then*

$$\begin{aligned} d(Z_t X_t) &= Z_t(\alpha_t - \beta_t Z_t X_t)\cdot(dW_t - \beta_t Z_t\, dt) \\ i.e. \quad d(Z_t X_t) &= \eta_t \cdot (dW_t - \beta_t Z_t\, dt) \end{aligned}$$

*for some process $\eta_t$.*

**Proof:** A straightforward application of Itô's formula.

□

Whereas the BGM approach shows how to define bond volatilities by *forward induction*, the MR approach directly constructs a set of measures under which forward LIBOR rates have the required volatility structure by *backward induction*. It is therefore convenient to introduce the following backward notation. Put

$$T_k^* = T_{N-k} \quad \text{so that} \quad T^* = T_0^* > T_1^* > \cdots > T_N^* = T_0$$

We start by working under a $T_N$–forward riskneutral measure $\mathbb{Q}^{T_N} = \mathbb{Q}^{T_0^*}$, together with a $\mathbb{Q}^{T_N}$–Brownian motion $W^{T_N} = W_t^{T_0^*}$. it is not necessary to construct this measure: we can assume that $\mathbb{Q}^{T_N}$ is the measure $\mathbb{P}$ which governs our model, and that $W^{T_N}$ is the $W_t$ which drives the economy. Ultimately, we will be able to specify all the dynamics under this measure, the *terminal measure*. Let $L(t, T_1^*) = L(t, T_{N-1})$ be a process which satisfies the SDE plus initial value

$$dL(t, T_1^*) = L(t, T_1^*)\lambda(t, T_1^*)\, dW_t^{T_N}$$
$$L(0, T_1^*) = \frac{P(0, T_1^*) - P(0, T_0^*)}{\delta_N P(0, T_0^*)}$$

This *defines* the forward LIBOR rate $L(t, T_1^*) = L(t, T_{N-1})$ in the MR model.
We now use this to define the forward LIBOR rate $L(t, T_2^*) = L(t, T_{N-2})$. To do so, we need to construct the forward riskneutral measure for maturity $T_2^*$. Under $\mathbb{Q}^{T_2^*}$, all the bond ratios $\frac{P(t,T_n^*)}{P(t,T_2^*)}$ are martingales. Now define the ratio

$$U_{N-n+1}(t, T_k) = \frac{P(t, T_k)}{P(t, T_n)} \quad \text{or, equivalently} \quad U_n(t, T_k^*) = \frac{P(t, T_k^*)}{P(t, T_{n-1}^*)}$$

and note that each $U_n(t, T_k^*)$ is required to be a martingale under the measure $\mathbb{Q}^{T_{n-1}^*}$ (which we must still construct). Further note that

$$U_2(t, T_k^*) = \frac{U_1(t, T_k^*)}{1 + \delta_N L(t, T_1^*)}$$

so that by the lemma,

$$dU_2(t, T_k^*) = \eta_{k,t} \cdot \left( dW^{T_N} - \frac{\delta_N L(t, T_1^*)}{1 + \delta_N L(t, T_1^*)} \lambda(t, T_1^*)\, dt \right)$$

for some process $\eta_{k,t}$ (whose exact nature is not important right now). In order for each $U_2(t, T_k^*)$ to be a martingale, it suffices to find a measure under which

$$W_t^{T_1^*} = W_t^{T_{N-1}} = W_t^{T_N} - \int_0^t \frac{\delta_N L(s, T_1^*)}{1 + \delta_N L(s, T_1^*)} \lambda(s, T_1^*)\, ds$$

is a Brownian motion. This is possible if we perform a Girsanov transformation from $\mathbb{Q}^{T_N} = \mathbb{Q}^{T_0^*}$ with kernel $\gamma(s, T_1^*) = \frac{\delta_N L(s,T_1^*)}{1+\delta_N L(s,T_1^*)} \lambda(s, T_1^*)$, i.e. if we define

$$\frac{d\mathbb{Q}^{T_1^*}}{d\mathbb{Q}^{T_0^*}} = \mathcal{E}_{T_1^*} \left( \int_0^{\cdot} \gamma(s, T_1^*)\, dW^{T_0^*} \right)$$

We now let $L(t, T_2^*)$ be a process which solves the SDE and initial condition

$$dL(t, T_2^*) = L(t, T_2^*)\lambda(t, T_2^*)\, dW_t^{T_1^*}$$
$$L(0, T_2^*) = \frac{P(0, T_2^*) - P(0, T_1^*)}{\delta_{N-1} P(0, T_1^*)}$$

We continue in this way: Suppose that we have already constructed the LIBOR rate processes $L(t, T_1^*), \ldots, L(t, T_n^*)$, for $n < N - 1$. Suppose further that this has been done so that each forward measure and Brownian motion has been specified, in particular that we have already constructed $\mathbb{Q}^{T_{n-1}^*}$ and $W_t^{T_{n-1}^*}$, and that $dL(t, T_n^*) = L(t, T_n^*)\lambda(t, T_n^*)\, dW_t^{T_{n-1}^*}$ under $\mathbb{Q}^{T_{n-1}^*}$. We must now construct a measure $\mathbb{Q}^{T_n^*}$ and an associated Brownian motion $W_t^{T_n^*}$. We require that each $U_{n+1}(t, T_k^*)$ is a $\mathbb{Q}^{T_n^*}$–martingale. Now

$$U_{n+1}(t, T_k^*) = \frac{U_n(t, T_k^*)}{1 + \delta_{N-n+1} L(t, T_n^*)}$$

Using the lemma, we see that

$$dU_{n+1}(t, T_k^*) = \eta_{k,t} \cdot \left( dW^{T_{n-1}^*} - \frac{\delta_{N-n+1} L(t, T_n^*)}{1 + \delta_{N-n+1} L(t, T_n^*)} \lambda(t, T_n^*)\, dt \right)$$

for some process $\eta_{k,t}$ (whose exact nature is not important right now). In order for each $U_{n+1}(t, T_k^*)$ to be a martingale, it suffices to find a measure under which

$$W_t^{T_n^*} = W_t^{T_{n-1}^*} - \int_0^t \frac{\delta_{N-n+1} L(s, T_n^*)}{1 + \delta_{N-n+1} L(s, T_n^*)} \lambda(s, T_n^*)\, ds$$

is a Brownian motion. This is possible if we perform a Girsanov transformation from $\mathbb{Q}^{T_{n-1}^*}$ with kernel $\gamma(s, T_n^*) = \frac{\delta_{N-n+1} L(s, T_n^*)}{1+\delta_{N-n+1} L(s, T_n^*)} \lambda(s, T_n^*)$, i.e. if we define

$$\frac{d\mathbb{Q}^{T_n^*}}{d\mathbb{Q}^{T_{n-1}^*}} = \mathcal{E}_{T_n^*}\left( \int_0^{\cdot} \gamma(s, T_n^*)\, dW_t^{T_{n-1}^*} \right)$$

We now let $L(t, T_{n+1}^*)$ be a process which solves the SDE and initial condition

$$dL(t, T_{n+1}^*) = L(t, T_{n+1}^*)\lambda(t, T_{n+1}^*)\, dW_t^{T_n^*}$$
$$L(0, T_{n+1}^*) = \frac{P(0, T_{n+1}^*) - P(0, T_n^*)}{\delta_{N-n} P(0, T_n^*)}$$

We have now constructed a sequence of processes $L(t, T_n)$ which are models of the forward LIBOR rates, with the desired volatilities. Since we also know the Girsanov kernels of each transformation, we can specify all LIBOR rate dynamics under the terminal measure. Inductively,

$$\begin{aligned}
dL(t, T_n^*) &= L(t, T_n^*)\lambda(t, T_n^*)\, dW_t^{T_{n-1}^*} \\
&= -L(t, T_n^*)\lambda(t, T_n^*)\gamma(t, T_{n-1}^*)\, dt + L(t, T_n^*)\lambda(t, T_n^*)\, dW_t^{T_{n-2}^*} \\
&= -L(t, T_n^*)\lambda(t, T_n^*)[\gamma(t, T_{n-1}^*) + \gamma(t, T_{n-2}^*)]\, dt + L(t, T_n^*)\lambda(t, T_n^*)\, dW_t^{T_{n-3}^*} \\
&= \ldots \\
&= -L(t, T_n^*)\lambda(t, T_n^*) \sum_{k=1}^{n-1} \gamma(t, T_{n-k}^*)\, dt + L(t, T_n^*)\lambda(t, T_n^*)\, dW_t^{T_0^*}
\end{aligned}$$

where

$$\gamma(t,T_k^*) = \frac{\delta_{N-k+1}L(t,T_k^*)}{1+\delta_{N-k+1}L(t,T_k^*)}\lambda(t,T_k^*)$$

and hence, when we translate from backwards time to ordinary time,

> The Musiela–Rutkowski forward LIBOR rate dynamics under the terminal measure $\mathbb{Q}^{T_N}$ are given by
>
> $$dL(t,T_n) = -L(t,T_n)\lambda(t,T_n)\sum_{k=n+1}^{N-1}\frac{\delta_{k+1}\lambda(t,T_k)L(t,T_k)}{1+\delta_{k+1}L(t,T_k)}\,dt + L(t,T_n)\lambda(t,T_n)\,dW_t^{T_N}$$
>
> This must be solved recursively: First find the solution for $L(t,T_{N-1})$. Once this has been found, find the solution for $L(t,T_{N-2})$. Note that the SDE for $L(t,T_{N-2})$ also contains $L(t,T_{N-1})$, but we've already found that. Then solve the SDE for $L(t,T_{N-3})$ (which contains $L(t,T_{n-1})$ and $L(t,T_{N-2})$; these have been determined). And so on...

It is therefore *possible* to find a model in which LIBOR rates have the required volatilities $\lambda(t,T_n)$. If these volatilities are deterministic, then each $L(t,T_n)$ will be lognormal under $\mathbb{Q}^{T_{n+1}}$. In that case, the Black formula for caps will be exact.

## 7.3 Jamshidian's Approach to Forward LIBOR

Like the Musiela–Rutkowski approach, Jamshidian(1997) does not require bond price dynamics, and models LIBOR rates for a discrete set of tenor dates $0 = T_{-1} < T_0 < T_1 < \cdots < T_N = T^*$ via a backward induction. But instead of working under the terminal measure, Jamshidian defines a *spot LIBOR measure*. This measure is obtained if we take as numéraire a certain portfolio of zero coupon bonds with unit initial value.

We begin by observing that the prices of discount bonds are not completely determined by the forward LIBOR rates. This is true at tenor dates, but if $t$ lies between tenor dates, e.g. $T_n < t < T_{n+1}$, then $P(t,T_{n+k}) = P(t,T_{n+1})\cdot\frac{1}{1+\delta_{n+2}L(t,T_{n+1})}\cdot\cdots\cdot\frac{1}{1+\delta_{n+k}L(t,T_{n+k-1})}$. Thus knowledge of the LIBOR rates is not enough — we also have to know the discount factor to the next tenor date (i.e. $P(t,T_{n+1})$). By working under the spot LIBOR measure, this problem can be circumvented.

Consider the following portfolio of discount bonds $X$. Its initial value is \$1.00. At all subsequent times, all wealth is invested in the next–to–mature bond. Thus at $t=0$, \$1.00 is invested in $P(t,T_0)$. At $T_0$, the payoff of these bonds is reinvested in $P(t,T_1)$ and at $T_1$, the payoff is reinvested in $P(t,T_2)$, etc. Thus at time $T_n$, the value of the portfolio is

$$\begin{aligned} X_{T_n} &= \frac{P(T_n,T_{n+1})}{P(0,T_0)\cdot P(T_0,T_1)\cdot\cdots\cdot P(T_n,P(T_{n+1})} \\ &= \text{value of } T_{n+1}\text{–bonds} \ \times\ \text{no. of } T_{n+1}\text{–bonds}\end{aligned}$$

An instant later, when $T_n \le t < T_{n+1}$, the value $X_t$ of the portfolio is simply

$$X_t = \frac{P(t,T_{n+1})}{P(0,T_0)\cdot P(T_0,T_1)\cdot\cdots\cdot P(T_n,P(T_{n+1})}$$

because the value of the $T_{n+1}$–bond has changed, but the number of $T_{n+1}$–bonds in the portfolio has not. Hence

$$X_t = P(t, T_{n(t)}) \cdot \prod_{k=0}^{n(t)} P^{-1}(T_{k-1}, T_k) \qquad (*)$$
$$\text{where } n(t) = \inf\{n : T_n > t\}$$

A *spot LIBOR measure* $\mathbb{Q}_X$ is obtained by taking $X_t$ as numéraire, so that each asset ratio process $\frac{P(t,T_n)}{X_t}$ is a $\mathbb{Q}_X$–martingale. The asset ratios can be written as

$$\begin{aligned}
\frac{P(t, T_{n+1})}{X_t} &= \frac{P(t, T_{n(t)}) \prod_{k=n(t)+1}^{n}(1 + \delta_k L(t, T_{k-1}))^{-1}}{P(t, T_{n(t)}) \prod_{k=0}^{n(t)}(1 + \delta_k L(T_{k-1}, T_{k-1})} \\
&= \prod_{k=0}^{n(t)}(1 + \delta_k L(T_{k-1}, T_{k-1}))^{-1} \prod_{k=n(t)+1}^{n} (1 + \delta_k L(t, T_{k-1}))^{-1} \\
&= \prod_{k=0}^{n} (1 + \delta_k L(t \wedge T_{k-1}, T_{k-1}))^{-1}
\end{aligned}$$

Hence the prices of the asset ratios are completely determined by the LIBOR processes. We now aim to describe the LIBOR rate dynamics under the spot LIBOR measure $\mathbb{Q}_X$, and that this requires knowledge only of the LIBOR rate volatilities (and not, say, bond or instantaneous forward rate volatilities as well). For the moment, assume that bond price dynamics are given by some Itô processes

$$\frac{dP(t, T_n)}{P(t, T_n)} = m(t, T_n)\, dt + S(t, T_n)\, dW_t$$

under the "real–world" probability measure $\mathbb{P}$. By definition of $X_t$ (i.e. by $(*)$), we see that

$$\frac{dX_t}{X_t} = m(t, T_{n(t)})\, dt + S(t, T_{n(t)})\, dW_t$$

Moreover, if we apply Itô's formula to $1 + \delta_{n+1} L(t, T_n) = \frac{P(t,T_n)}{P(t,T_{n+1})}$, we see that

$$\begin{aligned}
dL(t, T_n) &= \frac{P(t, T_n)}{\delta_{n+1} P(t, T_{n+1})} \Big[ \Big( m(t, T_n) - m(t, T_{n+1}) - (S(t, T_n) - S(t, T_{n+1}))S(t, T_{n+1}) \Big)\, dt \\
&\quad - \Big( S(t, T_n) - S(t, T_{n+1}) \Big)\, dW_t \Big] \\
&= \mu(t, T_n)\, dt + \zeta(t, T_n)\, dW_t
\end{aligned}$$

where

$$\begin{aligned}
\mu(t, T_n) &= \frac{P(t, T_n)}{\delta_{n+1} P(t, T_{n+1})} \Big( m(t, T_n) - m(t, T_{n+1}) \Big) - \zeta(t, T_n) S(t, T_{n+1}) \\
\zeta(t, T_n) &= \frac{P(t, T_n)}{\delta_{n+1} P(t, T_{n+1})} \Big( S(t, T_n) - S(t, T_{n+1}) \Big)
\end{aligned}$$

It follows that

$$S(t, T_{n(t)}) - S(t, T_{j+1}) = \sum_{k=n(t)}^{j} \frac{\delta_{k+1} \zeta(t, T_k)}{1 + \delta_{k+1} L(t, T_k)} \qquad (**)$$

for $j \geq n(t)$.

If $\gamma(t)$ is the Girsanov kernel for transforming $\mathbb{P}$ to $\mathbb{Q}_X$, i.e. if $\frac{d\mathbb{Q}_X}{d\mathbb{P}} = \mathcal{E}_{T^*}\left(\int_0^\cdot \gamma_t \, dW_t\right)$, then $\frac{P(t,T_n)}{X_t}$ has zero drift under $\mathbb{Q}_X$. But

$$d\frac{P(t,T_n)}{X_t} = \frac{P(t,T_n)}{X_t}\Big[\Big(m(t,T_n) - m(t,T_{n(t)}) - S(t,T_{n(t)}) \cdot (S(t,T_n) - S(t,T_{n(t)}))\Big)\, dt + \Big(S(t,T_n) - S(t,T_{n(t)})\Big)\, dW_t\Big]$$

Now in the Girsanov transformation, $(S(t,T_n) - S(t,T_{n(t)})) \cdot \gamma_t$ is added to the $\mathbb{P}$–drift to obtain the $\mathbb{Q}_X$–drift, which is zero, and so

$$m(t,T_n) - m(t,T_{n(t)}) - S(t,T_{n(t)}) \cdot (S(t,T_n) - S(t,T_{n(t)})) + (S(t,T_n) - S(t,T_{n(t)})) \cdot \gamma_t = 0$$

which yields

$$m(t,T_n) - m(t,T_{n+1}) = \Big(S(t,T_{n(t)}) - \gamma_t\Big) \cdot \Big(S(t,T_n) - S(t,T_{n(t)})\Big)$$

for $n = 0, \ldots, N$. It follows that

$$\begin{aligned} m(t,T_n) - m(t,T_{n+1}) &= \Big(m(t,T_n) - m(t,T_{n(t)})\Big) - \Big(m(t,T_{n+1}) - m(t,T_{n(t)})\Big) \\ &= \Big(S(t,T_{n(t)}) - \gamma_t\Big) \cdot \Big(S(t,T_n) - S(t,T_{n+1})\Big) \end{aligned}$$

Now multiply both sides of this equation by $\frac{P(t,T_n)}{\delta_{n+1}P(t,T_{n+1})}$ to obtain

$$\frac{P(t,T_n)}{\delta_{n+1}P(t,T_{n+1})}\Big(m(t,T_n) - m(t,T_{n+1})\Big) = \zeta(t,T_n)\Big(S(t,T_{n(t)}) - \gamma_t\Big)$$

Looking back to the definitions of $\mu$ and $\zeta$ in the dynamics of $L(t,T_n)$, we see that

$$\mu(t,T_n) = \zeta(t,T_n)\Big(S(t,T_{n(t)}) - \gamma_t - S(t,T_{n+1})\Big)$$

and hence

$$dL(t,T_n) = \zeta(t,T_n) \cdot \Big[\Big(S(t,T_{n(t)}) - S(t,T_{n+1}) - \gamma_t\Big)\, dt + dW_t\Big]$$

These are, of course, the $\mathbb{P}$–dynamics. To get the $\mathbb{Q}_X$–dynamics, we must add volatility $\times$ kernel $= \zeta_t \cdot \gamma_t$ to the drift to obtain

$$dL(t,T_n) = \zeta(t,T_n) \cdot \Big[\Big(S(t,T_{n(t)}) - S(t,T_{n+1})\Big)\, dt + dW_t^X\Big]$$

where $W_t^X = W_t - \int_0^t \gamma_u \, du$ is a $\mathbb{Q}_X$–Brownian motion. Finally, using $(**)$, we obtain

$$dL(t,T_n) = \sum_{k=n(t)}^{n} \frac{\delta_{k+1}\zeta(t,T_k) \cdot \zeta(t,T_n)}{1 + \delta_{k+1}L(t,T_k)}\, dt + \zeta(t,T_n) dW_t^X$$

These are the forward LIBOR rate dynamics under the spot LIBOR measure.

# 8 Appendix: Girsanov's Theorem

## 8.1 Motivation

When pricing contingent claims, we use risk-neutral valuation: The $t = 0$ price of a claim $X$ is the *risk–neutral* expectation of its discounted payoff.

$$X_0 = \mathbb{E}_{\mathbb{Q}}[\bar{X}]$$

The measure $\mathbb{Q}$ is not the same as the "real–world" measure $\mathbb{P}$ — we have to change the probability measure.

Suppose that we start with real–world asset dynamics, e.g. a GBM

$$\frac{dS_t}{S_t} = \mu\ dt + \sigma\ dW_t$$

on a filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $W_t$ is a $(\mathbb{F}, \mathbb{P})$–BM. There are two questions that concern us:

- What happens to the dynamics of $S_t$ when we change measures?
- How do we actually go about changing measures?

**Example 8.1** Earlier, we introduced *Wiener space* $(C[0,\infty), \mathcal{C}, \mathbb{P})$, which is a probability space with sample space $\Omega = C[0,\infty)$, the set of all continuous functions from $[0,\infty)$ to $\mathbb{R}$. This space comes equipped with a stochastic process $X$, the coordinate process, defined by $X_t(\omega) = \omega(t)$. (Remember that each $\omega \in C[0,\infty)$ is a function.) The $\sigma$–algebra $\mathcal{C}$ is generated by this process. Wiener space also comes equipped with a measure $\mathbb{P}$ — Wiener measure — which has the property that the coordinate process $X$ is a standard Brownian motion under $\mathbb{P}$.

Suppose now that we change the measure on $C[0,\infty)$, as follows: Let $\omega_0 : [0,\infty) \to \mathbb{R} : t \mapsto 0$ be the constant function with value 0. Let $\mathbb{Q} = \delta_{\omega_0}$ be the Dirac point mass, so that $\mathbb{Q}(A) = 1$ if $\omega_0 \in A$, and $\mathbb{Q}(A) = 0$ otherwise. Now what are the dynamics of the coordinate process $X$?

Clearly, $\mathbb{Q}(\{\omega : \forall t(\omega(t) = 0)\}) = \mathbb{Q}(\{\omega_0\}) = 1$, and hence, under $\mathbb{Q}$, the process $X$ is a.s. constant with value zero.

So though $X$ is a Brownian motion under $\mathbb{P}$, it doesn't remotely resemble a Brownian motion under $\mathbb{Q}$.

Note however, that $\mathbb{P}, \mathbb{Q}$ are not equivalent measures. It turns out that this sort of thing cannot happen when the measures are equivalent.

$\square$

Now recall that the stochastic integral $\int_0^T \theta_t\ dS_t$ has a very important financial interpretation: It is the gain made by trading the portfolio $\theta$. Stochastic integrals can only be defined for integrators $S$ which are semimartingales, i.e. it can only be defined if we can find a decomposition

$$S_t = S_0 + M_t + A_t$$

where $M$ is a local martingale, and $A$ a finite variation process. It is rather clear that changing measures usually destroys the martingale property — think of a game of coin tossing,

first under a measure where the coin is fair, and then change the measure to one where $\mathbb{P}(\text{Heads}) > 0.5$. So the decomposition above depends on the measure: The $\mathbb{P}$–local martingale may not be a $\mathbb{Q}$–local martingale. As a consequence, it is not obvious that a $\mathbb{P}$–semimartingale is necessarily a $\mathbb{Q}$–semimartingale, with the result that gains may be undefined in the risk–neutral world.

It turns out that, provided $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent measures*, all remains well. We now show why.

## 8.2 Girsanov's Theorem — General Statement

The following is adapted from *Stochastic Calculus: A Practical Introduction* by Rick Durrett, and *Stochastic Integration and Differential Equations*, by Philip Protter.

Because we have defined stochastic integrals only for continuous semimartingales, we will assume that our filtration only admits continuous martingales (up to a modification). This is the case, e.g., for the filtration generated by Brownian motion (by the Martingale Representation Theorem).

This is a good time to recall *Bayes' Theorem* for calculating conditional expectations when we change the measure: If $\mathbb{Q} << \mathbb{P}$, $\xi = \frac{d\mathbb{Q}}{d\mathbb{P}}$ and $\xi_t = \mathbb{E}_{\mathbb{P}}[\xi|\mathcal{F}_t]$, then

$$\xi_t \mathbb{E}_{\mathbb{Q}}[Z|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}}[Z\xi|\mathcal{F}_t]$$

We say that two measures $\mathbb{P}, \mathbb{Q}$ on a measurable space with filtration $\mathbb{F} = (\mathcal{F}_t)_t$ are *locally equivalent* iff $\mathbb{P}, \mathbb{Q}$ agree on $\mathcal{F}_t$, for every $t$. Let $\mathbb{P}_t, \mathbb{Q}_t$ be the restrictions of $\mathbb{P}, \mathbb{Q}$ to $\mathcal{F}_t$, respectively, so that $\mathbb{P}_t(A) = \mathbb{P}(A)$ for $A \in \mathcal{F}_t$, that

$$\mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}_t] = \mathbb{E}_{\mathbb{P}_t}[Z|\mathcal{F}_t]$$

etc.

Let $\alpha_t$ be a continuous version of the Radon–Nikodým process $\frac{d\mathbb{Q}_t}{d\mathbb{P}_t}$. Note that $\alpha$ is a $(\mathbb{F}, \mathbb{P})$–martingale: For if $A \in \mathcal{F}_s$, then $\mathbb{Q}_s(A) = \mathbb{Q}_t(A)$. Now $\mathbb{Q}_s(A) = \int_A \alpha_s \, d\mathbb{P}_s = \int_A \alpha_s \, d\mathbb{P}_t$, and $\mathbb{Q}_t(A) = \int_A \alpha_t \, d\mathbb{P}_t$, which implies that $\alpha_s$ is an $\mathcal{F}_s$–measurable variable with the property that

$$\int_A \alpha_s \, d\mathbb{P}_t = \int_A \alpha_t \, d\mathbb{P}_t \qquad \text{for all } A \in \mathcal{F}_s$$

and thus that $\mathbb{E}_{\mathbb{P}_t}[\alpha_t|\mathcal{F}_s] = \alpha_s$, by definition of conditional expectation.

**Lemma 8.2** *A process $Y_t$ is a $\mathbb{Q}$–(local) martingale iff $\alpha_t Y_t$ is a $\mathbb{P}$–(local) martingale.*

**Proof:** Fix $s < t$. Then by Bayes' Theorem,

$$\begin{aligned} & Y \text{ is } \mathbb{Q}\text{–martingale} \\ \Leftrightarrow \quad & \alpha_s Y_s = \alpha_s \mathbb{E}_{\mathbb{Q}_s}[Y_t|\mathcal{F}_s] = \mathbb{E}_{\mathbb{P}_t}[\alpha_t Y_t|\mathcal{F}_s] \\ \Leftrightarrow \quad & \alpha Y \text{ is a } \mathbb{P}\text{–martingale} \end{aligned}$$

This proves the result for martingales. Now use localization to obtain the result for local martingales.

⊣

**Exercises 8.3** Fill in the details in the above proof.

□

**Theorem 8.4** (Girsanov's Theorem) *If $Y$ is a $\mathbb{P}$–local martingale, then*

$$\tilde{Y} = Y - \alpha^{-1} \bullet [\alpha, Y]$$

*is a $\mathbb{Q}$–local martingale.*

**Proof:** Let $A_t = \int_0^t \alpha_s^{-1} d[\alpha, Y]_s$, and assume, for the moment, that this integral exists. We want $Y_t - A_t$ to be a $\mathbb{Q}$–local martingale, so it suffices to show that $\alpha_t(Y_t - A_t)$ is a $\mathbb{P}$–local martingale. By Ito's formula, using $[\alpha, A] = 0$ (because $A$ is of continuous and of bounded variation), we have

$$\begin{aligned} d\Big(\alpha_t(Y_t - A_t)\Big) &= \alpha_t(dY_t - dA_t) + (Y_t - A_t)\, d\alpha_t + d[\alpha, Y]_t \\ &= \alpha_t\, dY_t + (Y_t - A_t)\, d\alpha_t \end{aligned}$$

Since the integrators $Y, \alpha$ are $\mathbb{P}$–local martingales, so are the preceding stochastic integrals. Hence $\alpha_t(Y_t - A_t)$ is a $\mathbb{P}$–local martingale, as required.

We omit the (technical) proof that $A_t$ is well–defined for all $t$. It may be found in Durrett.

⊣

**Corollary 8.5** *If $Y$ is a $\mathbb{P}$–semimartingale, it is also a $\mathbb{Q}$–semimartingale.*

□

The following result is very important for mathematical finance.

**Theorem 8.6** *Let $Y$ be a semimartingale (under $\mathbb{P}, \mathbb{Q}$), and let $H$ be predictable. The quadratic variations $[Y]$ are the same under $\mathbb{P}, \mathbb{Q}$. The stochastic integrals $(H \bullet Y)$ are the same under $\mathbb{P}, \mathbb{Q}$*

**Proof:** We present only a heuristic outline of the proof: Recall that if $\mathbb{P} \sim \mathbb{Q}$, then convergence in $\mathbb{P}$–probability is equivalent to convergence in $\mathbb{Q}$–probability. Since the quadratic variation of $Y$ is a limit in probability of sums of the form $\sum_j (Y_{t_{j+1}} - Y_{t_j})^2$, the quadratic variations remain the same under $\mathbb{P}, \mathbb{Q}$.

As for the stochastic integrals: It is clear that if $A$ is a finite variation process, then $H \bullet A$ is the same under both measures, as it is defined pathwise (i.e. $\omega$–by–$\omega$) as a Stieltjes integral. It therefore remains to consider $H \bullet M$, where $M$ is a $\mathbb{P}$–local martingale. Any continuous local martingale can be made bounded, by stopping. In that case $(H \bullet M)_t$ is defined as an $L^2$ limit of $(H^n \bullet M)_t$, where the $H^n$ are simple processes. Now the simple integrals $H^n \bullet M$ are defined pathwise, and therefore coincide under both measures. Equivalence of measures ensures that the $L^2$–limits of the $H^n \bullet M$ are the same under both measures.

⊣

## 8.3 Exponentials

Recall the form of the Doléans exponential:

$$\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}[M]_t}$$

We know that

$$d\mathcal{E}(M)_t = \mathcal{E}(M)_t \, dM_t$$

and hence $\mathcal{E}(M)_t$ is a local martingale whenever $M_t$ is a local martingale. Doléans exponentials play a very important role in changes of measure: Recall that if $\mathbb{Q}$ is equivalent to $\mathbb{P}$, then $\alpha_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}_t}$ is a strictly positive $\mathbb{P}$–martingale. Suppose now that $D_t$ is an arbitrary continuous $\mathbb{P}$–local martingale. Define

$$M_t = \ln D_0 + \int_0^t D_s^{-1} \, dD_s$$

Then $M$ is a continuous local martingale also. Moreover, by Ito's formula,

$$d \ln D_t = D_t^{-1} \, dD_t - \frac{1}{2} \, D_t^{-2} d[D]_t = dM_t - \frac{1}{2} \, d[M]_t$$

from which it follows that

$$D_t = \mathcal{E}(M)_t$$

In fact, we have almost shown:

**Theorem 8.7** *If $D$ is a strictly positive continuous local martingale, there exists a unique continuous local martingale $M$ such that $D = \mathcal{E}(M)$.*

**Proof:** Existence has just been dealt with, so only uniqueness remains to be shown. If $M, N$ satisfy $\mathcal{E}(M) = D = \mathcal{E}(N)$, then taking logarithms yields $M - N = \frac{1}{2}([M] - [N])$. The lefthand side is a continuous local martingale, and the righthand side an FV process. It follows that $M - N$ is constant (as the only continuous local martingales of finite variation are the constants), and then that $M = N$.

⊣

**Example 8.8** Consider a one–stock Black Scholes model, with $\mathbb{P}$–dynamics

$$d\bar{S}_t = \bar{S}_t[(\mu - r) \, dt + \sigma \, dW_t]$$

over a finite horizon $[0, T]$. We want to find a measure $\mathbb{Q}$ under which $\bar{S}_t$ is a martingale for $t \leq T$. The way to do this is to find a continuous local martingale $\alpha_t$ for which $\alpha_t \bar{S}_t$ is a $\mathbb{P}$–martingale. If we can do that, we may define $\mathbb{Q}$ on $\mathcal{F}_T$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \alpha_T$.

So let's look for such an $\alpha$: Noting that $\alpha_0 = 1$, we can, by the martingale representation theorem, represent $\alpha$ as a stochastic integral:

$$\alpha_t = 1 + \int_0^t H_s \, dW_s$$

Now define $K_t = \frac{H_t}{\alpha_t}$, so that $\alpha_t = 1 + \int_0^t \alpha_s K_s \, dW_s$. Then $d\alpha_t = \alpha_t K_t \, dW_t$, and so $\alpha_t = \mathcal{E}(K \bullet W)_t$.

It follows that

$$\begin{aligned} d\Big(\alpha_t \bar{S}_t\Big) &= \alpha_t \bar{S}_t K_t \; dW_t + \alpha_t \bar{S}_t[(\mu - r)\; dt + \sigma \; dW_t] + \alpha_t K_t \bar{S}_t \sigma \; dt \\ &= \alpha_t \bar{S}_t \left[(K_t + \sigma)\; dW_t + (\mu - r + K_t \sigma)\; dt\right] \end{aligned}$$

So $\alpha_t \bar{S}_t$ will be a $\mathbb{P}$–local martingale provided that $K_t = \frac{r-\mu}{\sigma}$.

Thus we may define a risk–neutral measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\Big(\frac{r-\mu}{\sigma} \bullet W\Big)_T \qquad \text{on } \mathcal{F}_T$$

The only problem that remains is: Is it true that $\mathbb{Q}$ a probability measure, i.e. is $\mathbb{Q}(\Omega) = \mathbb{E}_{\mathbb{P}}[\mathcal{E}(K \bullet W)_T] = 1$?

Since $\mathcal{E}(K \bullet W)$ is always a non–negative local martingale, it is always a supermartingale. Hence the above question can be rephrased as: Is it true that $\mathcal{E}(K \bullet W)$ is a martingale?

□

**Exercises 8.9** Suppose that $M$ is supermartingale. Show that it is a martingale iff $\mathbb{E}X_t = \mathbb{E}X_0$ for all $t$.

□

As we have seen, a question that will be important in mathematical finance is the following:

Given that $M$ is a continuous local martingale, when is $\mathcal{E}(M)$ a genuine martingale?

As a matter of policy, we have often glossed over the technical differences between martingales and local martingales. Here, therefore, we will simply state two criteria that partially answer this question. See Protter for proofs.

**Theorem 8.10** (Kazamaki's criterion) *Suppose that $M$ is a continuous local martingale with the property that*

$$\sup_T \mathbb{E}[e^{\frac{1}{2}M_T}] < \infty \qquad \textit{where the } \sup \textit{ is over all bounded stopping times}$$

*Then $\mathcal{E}(M)$ is a UI martingale.*

**Theorem 8.11** (Novikov's criterion) *Let $M$ be a continuous local martingale, and assume that*

$$\mathbb{E}[e^{\frac{1}{2}[M]_\infty}] < \infty$$

*Then $\mathcal{E}(M)$ is a UI martingale.*

## 8.4 Girsanov's Theorem for Brownian Motion

**Theorem 8.12** (Girsanov's Theorem for Brownian Motion) *Suppose an $n$–dimensional process $Y$ has $\mathbb{P}$–dynamics*

$$dY_t = \mu_t \, dt + \sigma_t \, dW_t \qquad (t \leq T)$$

*where $W$ is a standard $d$–dimensional $\mathbb{P}$–Brownian motion, $\mu_t(\omega) \in \mathbb{R}^n, \sigma_t(\omega) \in \mathbb{R}^{n \times d}$. Let $\lambda_t(\omega) \in \mathbb{R}^d$ be predictable. Define a measure $\mathbb{Q}$ on $\mathcal{F}_T$ by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\lambda \bullet W)_T$$

*Assume that Novikov's condition holds:*

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T ||\lambda_s||^2 \, ds}\right] < \infty$$

*Then:*

*(i) $\mathbb{Q}$ is a probability measure on $\mathcal{F}_T$.*

*(ii) $\tilde{W}_t = W_t - \int_0^t \lambda_s \, ds$ is a $\mathbb{Q}$–Brownian motion.*

*(iii) The $\mathbb{Q}$–dynamics of $Y$ are given by*

$$dY_t = (\mu_t + \sigma_t \lambda_t) \, dt + \sigma_t \, d\tilde{W}_t$$

**Proof:** (i) We have $\mathbb{Q}(\Omega) = \mathbb{E}_{\mathbb{P}}[\mathcal{E}(\lambda \bullet W)_T] = \mathcal{E}(\lambda \bullet W)_0 = 1$, by Novikov's criterion. Hence $\mathbb{Q}$ is a probability measure.

(ii) Note that $\alpha_t = \frac{d\mathbb{Q}_t}{d\mathbb{P}}_t = \mathcal{E}(\lambda \bullet W)_t$ has $d\alpha_t = \alpha_t \lambda_t \, dW_t$, so that, by Girsanov's Theorem,

$$\tilde{W}_t = W_t - \int_0^t \alpha_s^{-1} \, d[\alpha, W]_s = W_t - \int_0^t \lambda_s \, ds$$

is a $\mathbb{Q}$–local martingale. Now since $\int_0^\bullet \lambda_t \, dt$ is a continuous FV–process, we see that

$$[\tilde{W}^i, \tilde{W}^j]_t = [W^i, W^j]_t = \delta_{ij} t$$

Hence $\tilde{W}$ is a continuous $\mathbb{Q}$–local martingale with the same covariance process as a $d$–dimensional Brownian motion. By Lévy's characterization, $\tilde{W}$ *is* a $\mathbb{Q}$–BM.

(iii) follows from the fact that $dY_t = \mu_t \, dt + \sigma_t \, dW_t$ and $dW_t = d\tilde{W}_t + \lambda_t \, dt$.

$\dashv$

**Remarks 8.13** This has important consequences: Suppose, under $\mathbb{P}$, we start with "real world" GBM dynamics

$$dS_t = S_t[\mu \, dt + \sigma \, dW_t] \qquad \text{i.e.} \qquad d\bar{S}_t = \bar{S}_t[(\mu - r) \, dt + \sigma \, dW_t]$$

Suppose we now construct a new measure $\mathbb{Q}$ as above. This *Girsanov transformation with kernel* $\lambda$ adds $\sigma\lambda$ to the drift of $\bar{S}$, but does not change the volatility:

$$d\bar{S}_t = \bar{S}_t[(\mu - r + \sigma\lambda) \, dt + \sigma \, d\tilde{W}_t]$$

For $\mathbb{Q}$ to be a risk–neutral measure, $\bar{S}_t$ must be driftless, i.e.

$$\sigma\lambda = r - \mu$$

With only one asset, this translates to $\lambda = -\frac{\mu - r}{\sigma}$, i.e. the Girsanov kernel is minus the market price of risk.

The fact that a Girsanov transformation does not affect the volatility is also important: It implies that we can use real–world observations to estimate risk–neutral world volatility.

□

# 9 Appendix: Correlated Brownian Motions

When many assets are available in the economy, it is unrealistic to assume that these are all driven by only one source of noise. It would be equally unrealistic, however, to assume that all are driven by separate, indpendent, Brownian motions. Thus it becomes necessary to generate multiple correlated Brownian motions.

Let's first consider a simpler case, where we are trying to generate *not* correlated Brownian motions, *just* correlated normal random variables, i.e. suppose that we want to generate mean zero normal random variables $X_1, \ldots, X_n$ with a specific covariance matrix $\Sigma = (\sigma_{ij})$. Here $\sigma_{ij} = \mathrm{Cov}(X_i, X_j)$.

You can check the following simple

**Fact:** If $(X_1, \ldots, X_n)$ is a random vector with covariance matrix $\Sigma$ and if $A$ is an $n \times n$–matrix, then the random vector

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = A \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

has covariance matrix $A\Sigma A^{tr}$.

Indeed, $\mathrm{Cov}(\mathbf{Y}) = \mathbb{E}[\mathbf{Y}\mathbf{Y}^{tr}] = A\mathbb{E}[\mathbf{X}\mathbf{X}^{tr}]A^{tr}$.

□

Covariance matrices are necessarily symmetric positive semidefinite, and it is known that symmetric positive semidefinite matrices have a *Cholesky decomposition*, which means that it is possible to find a (real) lower triangular matrix $C$ such that

$$\Sigma = CC^{tr}$$

Note that if $C$ is an arbitrary matrix, the $CC^{tr}$ is necessarily symmetric (obvious), and positive semidefinite: If $\vec{x}$ is a column vector, then $\vec{x}^{tr}C$ is a row vector, with length given by $||\vec{x}^{tr}C||^2 = (\vec{x}^{tr}C)(\vec{x}^{tr}C)^{tr} = \vec{x}^{tr}CC^{tr}\vec{x}$. Since the length of a vector is necessarily non–negative, $CC^{tr}$ is positive semidefinite.

Thus any matrix $A$ that can be written as $A = CC^{tr}$ is necessarily symmetric positive semidefinite. By the Cholesky decomposition, the reverse is also true. Indeed we can find a lower triangular $C$ which does the trick. There is no deep mathematics behind this — we

merely need to solve

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n_1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} c_{11} & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} & \dots & c_{n1} \\ 0 & c_{22} & \dots & c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & c_{nn} \end{pmatrix}$$

This system is easily solved: $c_{11}^2 = a_{11}$ gives us $c_{11}$. $c_{11}c_{21} = a_{12}$ now gives us $c_{21}$, etc.

There are fast algorithms available for calculating Cholesky decompositions.

Now suppose we are able to generate independent standard normal random variables $Z_1, \dots, Z_n$. These have the identity matrix as covariance matrix. Define a random vector $\mathbf{X} = C\mathbf{Z}$. Then the covariance matrix of $\mathbf{Z}$ is simply $CIC^{tr} = CC^{tr} = \Sigma$. Thus to get a vector $\mathbf{X}$ of mean zero normally distributed random variables with covariance matrix $\Sigma$, proceed as follows:

- Generate a vector $\mathbf{Z}$ of independent standard normal random variables (of the same dimension as $\mathbf{X}$).
- Find the Cholesky decomposition $\Sigma = CC^{tr}$ of the symmetric positive semidefinite matrix $\Sigma$.
- Put $\mathbf{X} = C\mathbf{Z}$

Note that if $\Sigma = (\sigma_{ij})$ is a covariance matrix, then the correlation matrix is given by

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$$

Clearly the correlation matrix is also symmetric.

Now to obtain *correlated Brownian motions* $W^1 1_t, \dots, W_t^1$, we can proceed in a similar way. But first: What exactly do we *mean* if we say two Brownian motions $W^1, W^2$ are correlated? Clearly this has meaning if we speak about *changes* in the processes. If $W^1, W^2$ are highly correlated, then we expect a positive change in $W^1$ to be accompanied by a positive change in $W^2$.

Now suppose that we have *independent* standard Brownian motions $B_t^1, B_t^n$. Consider a matrix $\Gamma = (\gamma_{ij})$ with the property that all the rows of $\Gamma$ have *unit length*. Define

$$\begin{pmatrix} W_t^1 \\ \vdots \\ W_t^n \end{pmatrix} = \Gamma \begin{pmatrix} B_t^1 \\ \vdots \\ B_t^n \end{pmatrix}$$

so that each $W_t^i = \sum_j \gamma_{ij} B_t^j$ is a linear combination of $B_t^j$'s. It follows that each $W_t^i$ is a continuous local martingale. Now

$$\begin{aligned} {}_t &= \left(\sum_j \gamma_{ij}\right)\left(\sum_k \gamma_{ik}\right)[B^j, B^k]_t \\ &= \sum_j \gamma_{ij}^2 t \\ &= t \end{aligned}$$

because $[B^j, B^k]_t = \delta_{jk} t$ and $\sum_j \gamma_{ij}^2 = 1$. Hence, by Lévy's Characterization, each $W_t^i$ is a Brownian motion. Now

$$
\begin{aligned}
t &= \sum_{k,l} \gamma_{ik} \gamma_{jl} \delta_{kl} t \\
&= (\Gamma \Gamma^{tr})_{ij} t
\end{aligned}
$$

which we may also write as

$$dW_t^i \, dW_t^j = (\Gamma \Gamma^{tr})_{ij} \, dt$$

Hence

$$\mathbb{E}[W_t^i W_t^j] = \mathbb{E}[\, [W^i, W^j]_t] = (\Gamma \Gamma^{tr})_{ij} t$$

Thus $W^i, W^j$ are correlated Brownian motions, and the correlation between $W_t^i$ and $W_t^j$ is simply $(\Gamma \Gamma^{tr})_{ij}$, independent of $t$ (because the variance of each $W_t^i$ is just $t$).

Note that if $\Sigma, \rho$ are, respectively, the covariance and correlation matrix of $(W_t^1, \ldots, W_t^n)$, then $\Sigma = \rho t$. Hence $\rho$ is also symmetric positive semidefinite, and thus has a Cholesky decomposition $\rho = \Gamma \Gamma^{tr}$.

Further note that not every symmetric positive semidefinite matrix can be the correlation matrix of some multidimensional Brownian motion: Since the correlation of a random variable with itself is 1, it is necessary that a correlation matrix has 1's down the diagonal. This, in turn, implies that the Cholesky decomposition matrix $\Gamma$ will have row vectors of unit length.

Hence, to create correlated Brownian motions with correlation matrix $\rho$, proceed as follows:

- Find the Cholesky decomposition $\rho = \Gamma \Gamma^{tr}$. $\Gamma$ will have rows of unit length.
- Define $\mathbf{W} = \Gamma \mathbf{B}$, where $\mathbf{B}$ is a multidimensional standard Brownian motion (with independent component processes). $\mathbf{W}$ will be a multidimensional Brownian motion with correlation matrix $\rho$.

One final remark about differential notation: Since $\Gamma \Gamma^{tr} = \rho$, and since $dW_t^i \, dW_t^j = (\Gamma \Gamma^{tr})_{ij} \, dt$, we have

$$d[W^i, W^j]_t = dW_t^i \, dW_t^j = \rho_{ij} \, dt$$

**Example 9.1** To create two correlated Brownian motions $W_t^1, W_t^2$ with correlation $\rho$ (a number, not a matrix), proceed as follows: The correlation matrix is

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Its Cholesky decomposition is found by solving

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \qquad (= \Gamma \Gamma^{tr})$$

for $a, b, c$. (Recall that $\Gamma$ is lower triangular.) Thus $a = 1, b = \rho, c = \sqrt{1 - \rho^2}$. Gratifyingly, the rows of $\Gamma$ are seen to possess unit length.

Finally, if $B_t^1, B_t^2$ are standard independent Brownian motions, then

$$
\begin{aligned}
W_t^1 &= B_t^1 \\
W_t^2 &= \rho B_t^1 + \sqrt{1 - \rho^2} B_t^2
\end{aligned}
$$

are Brownian motions with correlation $\rho$.

□

**Example 9.2** Suppose we have asset dynamics

$$\begin{pmatrix} dS_t^1 \\ dS_t^2 \end{pmatrix} = \begin{pmatrix} 0.3S_t^1 \\ 0.2S_t^2 \end{pmatrix} dt + \begin{pmatrix} 0.1S_t^1 & 0.4S_t^1 \\ 0.4S_t^2 & 0.3S_t^2 \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}$$

where $W_t^1, W_t^2$ are independent Brownian motions. Here each asset is driven by two sources of noise. It may be convenient to rewrite the dynamics in a decoupled fashion:

$$\begin{aligned} dS_t^1 &= 0.3S_t^1 \; dt + \hat{\sigma}_1 S_t^1 \; d\hat{W}_t^1 \\ dS_t^2 &= 0.2S_t^2 \; dt + \hat{\sigma}_2 S_t^2 \; d\hat{W}_t^2 \end{aligned}$$

where $\hat{W}_t^1, \hat{W}_t^2$ are *correlated* Brownian motions. This *may* be simpler, because each asset is now driven by only one source of noise.

The two things that we need to know are:

(i) What are the volatilities $\hat{\sigma}_1, \hat{\sigma}_2$?

(ii) What is the correlation $\rho$ between $\hat{W}_t^1$ and $\hat{W}_t^2$?

Clearly, we must have

$$\begin{aligned} \hat{\sigma}_1 \; d\hat{W}_t^1 &= 0.1 \; dW_t^1 + 0.4 \; dW_t^2 \\ \hat{\sigma}_2 \; d\hat{W}_t^2 &= 0.4 \; dW_t^1 + 0.3 \; dW_t^2 \end{aligned}$$

Looking at the covariance processes, we must have

$$\begin{aligned} \hat{\sigma}_1^2 \; dt &= (0.1^2 + 0.4^2) \; dt \\ \hat{\sigma}_2^2 \; dt &= (0.4^2 + 0.3^2) \; dt \\ \hat{\sigma}_1 \hat{\sigma}_2 \rho \; dt &= (0.1 \times 0.4 + 0.4 \times 0.3) \; dt \end{aligned}$$

which are three equations in 3 unknowns, easily solved for $\hat{\sigma}_1, \hat{\sigma}_2, \rho$:

$$\begin{aligned} \hat{\sigma}_1 &= ||(0.1, 0.4)|| \\ \hat{\sigma}_2 &= ||(0.4, 0.3)|| \\ \rho &= \frac{(0.1, 0.4) \cdot (0.4, 0.3)}{||(0.1, 0.4)|| \cdot ||(0.4, 0.3)||} \end{aligned}$$

Note that the vectors on the right can all be read off the volatility matrix.

Thus

$$\begin{aligned} \hat{W}_t^1 &= \frac{(0.1, 0.4) \cdot (W_t^1, W_t^2)}{||(0.1, 0.4)||} \\ \hat{W}_t^2 &= \frac{(0.4, 0.3) \cdot (W_t^1, W_t^2)}{||(0.4, 0.3)||} \end{aligned}$$

It is clear that $\hat{W}^1, \hat{W}^2$ are continuous martingales. Moreover

$$[\hat{W}^1]_t = t = [\hat{W}^2]_t$$

so that $\hat{W}^1, \hat{W}^2$ are indeed Brownian motions (by Lévy's Characterization). Furthermore,

$$[\hat{W}^1, \hat{W}^2]_t = \rho \; dt$$

as expected.

□

The above example can be generalized:

**Proposition 9.3** *Give a multidimensional SDE $dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$, i.e.*

$$\begin{pmatrix} dX_t^1 \\ \vdots \\ dX_t^n \end{pmatrix} = \begin{pmatrix} b^1(t, X_t) \\ \vdots \\ b^n(t, X) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(t, X_t) & \dots & \sigma_{1m}(t, X_t) \\ \vdots & \vdots & \vdots \\ \sigma_{n1}(t, X_t) & \dots & \sigma_{nm}(t, X_t) \end{pmatrix} \begin{pmatrix} dW_t^1 \\ \vdots \\ dW_t^m \end{pmatrix}$$

*where $W_t = (W_t^1, \dots, W^m)_t$ is a standard $m$–dimensional Brownian motion. Let $\sigma_i$ be the $i^{th}$ row of the matrix $\sigma$. Define*

$$\hat{W}_t^i = \frac{\sigma_i \cdot W_t}{||\sigma_i||} \qquad \text{for } i = 1, \dots, n$$

*Then (by Lévy's Characterization) the $\hat{W}_t^i$ are $n$ correlated Brownian motions, with correlation*

$$\rho_{ij} = \frac{\sigma_i \cdot \sigma_j}{||\sigma_i|| \cdot ||\sigma_j||}$$

*and we have dynamics*

$$dX_t^i = b^i(t, X_t)\, dt + ||\sigma_i(t, X_t)||\, d\hat{W}_t^i \qquad \text{for } i = 1, \dots, n$$

*Here each $X^i$ is driven by only one source of noise.*

□

Thus the "volatility" of a one–dimensional process of the form

$$dX_t = \mu\, dt + \sigma_1\, dW^1 + \dots + \sigma_n\, dW^n$$

is

$$\hat{\sigma} = ||(\sigma_1, \dots, \sigma_n)|| = \sqrt{\sigma_1^2 + \dots + \sigma_n^2}$$

What happens to the Black–Scholes PDE when we have correlated Brownian motions? Recall that this is

$$\frac{\partial C}{\partial t} + \sum_n r S^n \frac{\partial C}{\partial S^n} + \frac{1}{2} \sum_{n,m} (\sigma \sigma^{tr}) S^n S^m \frac{\partial^2 C}{\partial S^n \partial S^m} - rC = 0$$

Note that $(\sigma\sigma^{tr})_{nm}$ is just $\sigma_n \cdot \sigma_m$, the inner product of the $n^{th}$ and $m^{th}$ rows of $\sigma$. We have seen that

$$\rho_{nm} = \frac{\sigma_n \cdot \sigma_m}{||\sigma_n|| ||\sigma_m||} = \frac{\sigma_n \cdot \sigma_m}{\hat{\sigma}_n \hat{\sigma}_m}$$

and thus we obtain

$$\frac{\partial C}{\partial t} + \sum_n r S^n \frac{\partial C}{\partial S^n} + \frac{1}{2} \sum_{n,m} \rho_{nm} \hat{\sigma}_n \hat{\sigma}_m S^n S^m \frac{\partial^2 C}{\partial S^n \partial S^m} - rC = 0$$

where $\hat{\sigma}_n$ is the volatility of $S^n$.

# 10 Exercises

1. An endowment option $X$ is a very long term European call option. Typically,

   - At issue, the initial strike $K_0$ is set to approximately 50% of the current stock price.
   - The options are inflation and dividend protected:
     - The strike price increases at the short term riskless rate.
     - The strike price is decreased by the size of the dividend each time a dividend is paid.
   - The payoff at expiry $T$ is $X_T = (S_T - K_T)^+$.

   We will make the simplifying assumption that the stock pays no dividends. This can be accomplished by regarding the stock price as the theoretical price of a mutual fund which starts off at one share, and reinvests all dividends in that share. We have, in the risk–neutral world,

   $$dS_t = r_t S_t\ dt + \sigma_t S_t\ dW_t \quad dA_t = r_t A_t\ dt$$

   where $S$ is the share and $A$ the money market account (with $A_0 = 1$). Clearly $K_t = K_0 A_t$. By changing the numéraire to $A_t$, show that, when the volatility $\sigma_t$ is deterministic,

   $$X_0 = S_0 N(d_+) - K_0 N(d_-)$$

   where

   $$d_\pm = \frac{\ln \frac{S_0}{K_0} \pm \frac{1}{2}\sigma_{av}^2 T}{\sigma_{av}\sqrt{T}} \qquad \text{and} \qquad \sigma_{av}^2 = \frac{1}{T}\int_0^T \sigma_t^2\ dt$$

2. Use the change–of–numéraire technique to show how to calculate the value of an option which pays the minimum of two assets $S^1, S^2$. Assume that the "real world" dynamics of the assets are Itô diffusions of the form

   $$dS_t^i = S_t^i[\mu_i\ dt + \hat{\sigma}_i\ d\hat{W}_t^i]$$

   where $\mu_i, \sigma_i$ are constants, and that the correlation of returns is a constant $\rho$. Further assume that $S_t^i$ has a continuously paid dividend with constant dividend yield $q^i$.

3. Consider a European call $C$ on share $S$ traded on FTSE. $S_t$ and $C$ are priced in pounds, but the strike of the call is in dollars. Initially, the option is at–the–money. The dollar strike does not change, but because exchange rates are not fixed, the pound strike does. Let $X_t$ be the $\frac{\text{dollar}}{\text{pound}}$-rate, $Y_t$ the $\frac{\text{pound}}{\text{dollar}}$–rate. Assume dynamics

   $$\begin{aligned} dS_t &= \alpha_S S_t\ dt + \delta_S S_t\ dW_t^S \\ dX_t &= \alpha_X X_t\ dt + \delta_X X_t\ dW_t^X \\ dY_t &= \alpha_Y Y_t\ dt + \delta_Y Y_t\ dW_t^Y \end{aligned}$$

   where $W^S, W^X, W^Y$ are correlated Brownian motions.

   3.1 Apply Ito's formula to show

   $$dY_t = \alpha_Y Y_t\ dt + \delta_Y Y_t(-dW_t^X) \qquad \alpha_Y = -\alpha_X + \delta_X^2, \quad \delta_Y = \delta_X$$

3.2 Let $\rho$ be the correlation between $W^S$ and $W^X$. Let $W_t = (W_t^1, W_t^2)$ be a two–dimensional standard Brownian motion, and rewrite the above dynamics

$$\begin{aligned} dS_t &= \alpha_S S_t \, dt + S_t \sigma_S \, dW_t \\ dX_t &= \alpha_X X_t \, dt + X_t \sigma_X \, dW_t \\ dY_t &= \alpha_Y Y_t \, dt + Y_t \sigma_Y \, dW_t \end{aligned}$$

Show that we must have

$$\begin{aligned} ||\sigma_X||^2 = \delta_X^2 \quad ||\sigma_Y||^2 = \delta_Y^2 \quad ||\sigma_S||^2 = \delta_S^2 \\ \sigma_X \cdot \sigma_S = \rho\delta_X\delta_S \quad \sigma_Y \cdot \sigma_S = -\rho\delta_X\delta_S \end{aligned}$$

3.3 The initial pound strike is $K_0^p = S_0$, and the initial dollar strike is $K^d = S_0 X_0$ (at–the–money), which remains fixed. At maturity, the pound strike is $K_T = K^d Y_T$. Define $S_t^d = S_t X_t$ to be the dollar price of $S$ at time $t$. Show that

$$dS_t^d = S_t^d[\alpha_S + \alpha_X + \sigma_S \cdot \sigma_X] \, dt + S_t^d(\sigma_S + \sigma_X) \, dW_t$$

3.4 Now convert this to a system with a one–dimensional Brownian motion $V_t$:

$$dS_t^d = S_t^d[\alpha_S + \alpha_X + \sigma_S \cdot \sigma_X] \, dt + S_t^d \delta_{S^d} \, dV_t$$

where

$$\delta_{S^d}^2 = ||\sigma_X + \sigma_S||^2 = (\delta_X^2 + \delta_S^2 + 2\rho\delta_X\delta_S)$$

3.5 Now we have a plain vanilla call on an asset $S^d$ with (fixed) strike $K^d$. Find the dollar price $C_t^d$ of this option:

$$C_t^d = S^d N(d_+) - e^{-r_d(T-t)} K^d N(d_-)$$

where where $r_d$ is the riskless *dollar* rate, and

$$d_\pm = \frac{\ln \frac{S_t^d}{S_0^d} + (r_d \pm \frac{1}{2}\delta_{S^d}^2)(T-t)}{\delta_{S^d}\sqrt{T-t}}$$

3.6 Conclude that the pound price of the option is

$$C_t = S_t N(d_+) - e^{-r_d(T-t)} \frac{S_0 X_0}{X_t} N(d_-) \qquad d_\pm = \frac{\ln \frac{S_t X_t}{S_0 X_0} + (r_d \pm \frac{1}{2}(\delta_X^2 + \delta_S^2 + 2\rho\delta_X\delta_S))(T-t)}{\sqrt{(\delta_X^2 + \delta_S^2 + 2\rho\delta_X\delta_S)(T-t)}}$$

3.7 If we had tried to price the option directly in pounds, we would have had (explain this)

$$C_T = (S_T - S_0(Y_T/Y_0))^+$$

Very naturally, we would have considered the numeraire $Y_t$. This would have been a mistake, for although $Y_t$ is a traded asset (namely the pound price of a dollar note), this is not a non–dividend paying asset: $Y_t$ has a continuous dividend yield equal to the riskless dollar rate $r_d$. Thus discounted $Y_t$ is not a $\mathbb{Q}$–martingale. Instead, therefore,

consider the process $\hat{Y}_t = Y_t e^{r_d t}$. (i.e. all dividends = interest reinvested in the dollar money market account). Show that

$$C_t = \hat{Y}_t \mathbb{E}_{\hat{\mathbb{Q}}}[(\hat{S}_t - K')^+ | \mathcal{F}_t]$$

where $K' = e^{-r_d T} S_0 / Y_0$ and $\mathbb{Q}$ is the equivalent martingale measure associated with $\hat{Y}$.

3.8 Find the $\hat{\mathbb{Q}}$–dynamics of $\hat{S}_t$ (with a two–dimensional standard Brownian motion).

3.9 Convert this to $S_t$–dynamics with a one–dimensional Brownian motion.

3.10 Hence show that

$$C_t \hat{Y}_t [\hat{S}_t N(d_+) - K' N(d_-)]$$

where

$$d_\pm = \frac{\ln \frac{\hat{S}_t}{K'} \pm \frac{1}{2}\hat{\delta}^2 (T-t)}{\hat{\delta}\sqrt{T-t}}$$

and $\hat{\delta} = ||\sigma_S - \sigma_Y|| = \sqrt{\delta_S^2 + \delta_Y^2 + 2\rho\delta_S\delta_Y}$.

3.11 Finally show that this coincides with the formula obtained earlier.
In this case, you see that it is slightly easier to value the option in dollars than it is in pounds.

4. Suppose the bond price dynamics are given by

$$dp(t,T) = p(t,T) M(t,T)\, dt + p(t,T) v(t,T)\, dW_t$$

Show that in that case the forward rate dynamics are given by

$$df(t,T) = \alpha(t,T)\, dt + \sigma(t,T)\, dW_t$$

where

$$\alpha(t,T) = v_T(t,T) v(t,T) - m_T(t,T) \qquad \sigma(t,T) = -v_T(t,T)$$

[Hint: Apply Ito's formula to $\ln p(t,T)$, write this in integrated form, and differentiate with respect to $T$.]

5. Let $\{y(0,T) : T \geq 0\}$ denote the zero–coupon yield curve at $t = 0$. Assume that, apart from the zero coupon bonds, we also have exactly one fixed coupon bond for every maturity $T$. enote the yield–to–maturity of the fixed coupon bond by $y_M(0,T)$. We now have 3 curves to consider, the forward rate curve $f(0,T)$, the zero yield curve $y(0,T)$ and the coupon yield curve $y_M(0,T)$.

5.1 Show that $f(0,T) = y(0,T) + T\dfrac{\partial y(0,T)}{\partial T}$

5.2 Assume that the zero yield curve is an increasing function of $T$. Show that in that case

$$y_M(0,T) \leq y(0,T) \leq f(0,T)$$

for all $T$. Show that the inequalities are reversed if the zero yield curve is decreasing. Explain this phenomenon in terms of simple economics.

5.3 Yield curves can be both upward and downward sloping. Can this be true for bond price curves $p(0,T)$?

6. In the Cox–Ingersoll–Ross model, the risk–neutral short rate dynamics assumed are

$$dr_t = (b - ar_t)\,dt + \sigma\sqrt{r_t}\,dW_t, \qquad a, b, \sigma, r_0 > 0$$

6.1 Explain (heuristically) why this process is mean–reverting and non–negative.

6.2 This is an affine short rate model. By plugging $p(t,T) = e^{A(t,T)-B(t,T)r_t}$ into the term structure PDE, show that we obtain two coupled ODE's

$$\begin{aligned} B_t &= aB + \frac{1}{2}\sigma^2 B^2 - 1 \qquad & B(T,T) &= 0 \\ A_t &= bB & A(T,T) &= 0 \end{aligned}$$

6.3 To solve the Riccati equation for $B$, try a solution of the form

$$B(t,T) = \frac{X(t)}{cX(t) + d}$$

Choose $c$ to ensure that $a + \frac{1}{2}\sigma^2 - c^2 = 0$. Show that we then obtain a order linear differential equation

$$X_t + \kappa X = -d \qquad \text{where } \kappa = -a + 2c = \sqrt{a^2 + 2\sigma^2}$$

6.4 Solve the ODE to obtain

$$X(t) = \frac{d}{\kappa}[e^{\kappa(T-t)} - 1]$$

6.5 Hence show that

$$B(t,T) = \frac{2(e^{\kappa(T-t)} - 1)}{2\kappa + (a+\kappa)(e^{\kappa(T-t)} - 1)} \qquad \text{where } \kappa = \sqrt{a^2 + 2\sigma^2}$$

6.6 Verify by differentiation that

$$A(t,T) = \frac{2b}{\sigma^2} \ln\left[\frac{2\kappa e^{\frac{1}{2}(a+\kappa)(T-t)}}{2\kappa + (a+\kappa)(e^{\kappa(T-t)} - 1)}\right]$$

7. 7.1 Show that the Hull–White model $dr = (\theta(t) - ar)\,dt + \sigma\,dW_t$ is obtained if one starts with a HJM model given by

$$df(t,T) = \alpha(t,T)\,dt + \sigma e^{-a(T-t)}\,dW_t$$

Hence compute the function $\theta(t)$ which will make the short rate model fit the initial term structure:

$$\theta(t) = f_T^*(0,t) + af^*(0,t) + \frac{\sigma^2}{a}[1 - e^{-2at}]$$

where $\{f^*(0,T) : T \geq 0\}$ is the observed term structure of forward rates. It follows that the Hull–White model can also be fitted to any initial term structure. What is the distribution of the forward rate $f(t,T)$?

7.2 Show that bond prices in the Hull–White model, fitted to the initial term structure, are given by

$$p(t,T) = \frac{p(0,T)}{p(0,t)} \exp\left( f(0,t)B(t,T) - \frac{\sigma^2}{4a} B^2(t,T)(1 - e^{-2at}) - B(t,T)r_t \right)$$

where $B(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}]$.
[Hint: The Hull–White model is an affine term structure model, i.e. $p(t,T) = e^{A(t,T) - B(t,T)r_t}$. $B(t,T)$ is readily calculated. We can now find

$$A(t,T) = \int_t^T -\theta(u)B(u,T)\, du + \frac{1}{2}\sigma^2 B^2(0,t)$$

where $\theta(t)$ is as in (a), i.e. $b(u) = e^{-au}\frac{d}{du}x(u)e^{au}$, where $x(t) = e^{-at}r_0 + \int_0^t e^{-a(t-u)}\, du$. Integrating by parts leads to

$$\begin{aligned} A(t,T) &= f(0,t)B(t,T) + \ln \frac{p(0,T)}{p(0,t)} \\ &\quad + \frac{\sigma^2}{2}\left[ \int_t^T b^2(u,T) - B^2(0,u)\, du + B^2(0,t)B(t,T) \right] \end{aligned}$$

which simplifies to give the required result.]

7.3 Show that the Hull–White price of a call $C$ with strike $K$ and maturity $T$ on a bond $p(0,S)$ (where $S > T$) is given by

$$p(0,S)N(d_+) - Kp(0,T)N(d_-)$$

where

$$d_{\pm} = \frac{\ln \frac{p(0,S)}{Kp(0,T)} \pm \frac{1}{2}\sigma_{av}^2 T}{\sigma_{av}\sqrt{T}}$$

and where

$$\sigma_{av}^2 T = \int_0^T \frac{\sigma^2}{a^2}(e^{-aS} - e^{-aT})^2 e^{2at}\, dt = \frac{\sigma^2}{2a^3}(1 - e^{-a(S-T)})^2(1 - e^{-2aT})$$

8. Consider the domestic and the foreign bond market, with bond prices denoted by $p_d(t,T)$ and $p_f(t,T)$ respectively. Take as given a standard HJM model for the domestic forward rates $f_d(t,T)$

$$df_d(t,T) = \alpha_d(t,T)\, dt + \sigma_d(t,T)\, dW_t$$

where $W_t$ is a multidimensional Brownian motion under the *domestic* martingale measure $\mathbb{Q}$. The foreign forward rates are denoted by $f_f(t,T)$, and their $\mathbb{Q}$–dynamics are given by

$$df_f(t,T) = \alpha_f(t,T)\, dt + \sigma_f(t,T)\, dW_t$$

Note that the same Brownian motion drives both bond markets. The exchange rate $X$ (in units of domestic currency per unit of foreign currency) has $\mathbb{Q}$–dynamics

$$dX_t = \mu(t)X(t)\, dt + \sigma_X(t)X(t)\, dW_t$$

Show that under the domestic martingale measure the foreign forward rates satisfy the modified HJM drift condition

$$\alpha_f(t,T) = \sigma_f(t,T)\left[\int_t^T \sigma_f^{\text{tr}}(t,s)\,ds - \sigma_X^{\text{tr}}(t)\right]$$

9. A common implementation of the HJM framework uses the following forward rate dynamics:

$$df(t,T) = \alpha(t,T)\,dt + (\sigma_1, \sigma_2 e^{-\frac{\lambda}{2}(T-t)}) \cdot (dW_1(t), dW_2(t))$$

where $\sigma_1, \sigma_2, \lambda$ are non–negative constants, $W_1, W_2$ are independent $\mathbb{Q}$–Brownian motions, and $\mathbb{Q}$ is the equivalent risk–neutral measure.

This is a two–factor model. The first factor $W_1(t)$ can be interpreted as a source of noise that lasts a long time, affecting all maturities equally. The second factor $W_2(t)$ affects short maturity forward rates more than the long term rates (why?), and thus adds some extra volatility to the short term rates.

9.1 Show that the HJM drift conditions imply that

$$\alpha(t,T) = \sigma_1^2(T-t) - \frac{2\sigma_2^2}{\lambda} e^{-\frac{\lambda}{2}(T-t)}(e^{-\frac{\lambda}{2}(T-t)} - 1)$$

9.2 Hence show that

$$\begin{aligned} f(t,T) = {} & f(0,T) + \sigma_1^2 t(T - t/2) - 2(\sigma_2/\lambda)^2[e^{-\lambda T}(e^{\lambda t} - 1) - 2e^{-(\lambda/2)T}(e^{(\lambda/2)t} - 1)] \\ & + \sigma_1 W_1(t) + \sigma_2 \int_0^t e^{-(\lambda/2)(T-u)}\,dW_2(u) \end{aligned}$$

9.3 Show that the spot rate follows the process

$$\begin{aligned} r(t) = {} & f(0,t) + \frac{1}{2}\sigma_1^2 t^2 - 2(\sigma_2/\lambda)^2[1 - e^{-(\lambda/2)t}]^2 \\ & + \sigma_1 W_1(t) + \sigma_2 e^{-(\lambda/2)t}\int_0^t e^{(\lambda/2)u}\,dW_2(u) \end{aligned}$$

9.4 Is the short rate a Markov process, a Gaussian process, a stationary process? Explain.

9.5 Calculate the price $C(t)$ of a call option on the zero coupon bond $p(t,T)$. Assume that the option has strike $K$ and expiry $\tau$, where $t \le \tau \le T$.
[Hint: Let $p(t,\tau)$ be the numeraire. You know the HJM dynamics of zero coupon bonds under $\mathbb{Q}$, so the dynamics of $p(t,T)/p(t,\tau)$ under the EMM for $p(t,\tau)$ should be easy to find. Of course, something is going to be lognormal. Now use the general option pricing formula.]

9.6 As a check, assume that $\sigma_1 = 0.2, \sigma_2 = 0.3$ and $\lambda = 2$. Calculate the value of a call option on a two–year zero coupon bond with strike 0.9 and expiry 1 year. Today's prices are $P(0,1) = 0.9, P(0,2) = 0.81$. I get 0.076 (but I could be wrong, of course).

10. Consider a convertible bond $X$ which, at $T_0$, allows the owner to convert the bond to c shares $S$ of common stock. The bond is a zero coupon bond with face value 1.00 and maturity $T_1 > T_0$. The aim of this problem is to find the price the convertible bond at

some future date $t \le T_0$. We will model the short rate using Ho–Lee dynamics. Initially, the (instantaneous) forward rate curve is flat with $f(0,T) = r_0$ for all maturities $T$.

We work under a risk-neutral measure $\mathbb{Q}$ where the share has dynamics

$$dS_t = r(t)S_t\,dt + \sigma_S S_t\,dW_t$$

and the short rate has dynamics

$$dr(t) = \theta(t)\,dt + \sigma_r\,dW_t$$

Here $W_t$ is a two–dimensional $\mathbb{Q}$–Brownian motion, and $\sigma_S, \sigma_r$ are constant vectors.

10.1 Let $p(t,T)$ be a non–convertible zero coupon bond with face value 1.00 and maturity $T$ years. Calculate the observed term structure of bond prices $\{p^*(0,T) : T \ge 0\}$.

10.2 Let $\hat{\mathbb{Q}}$ be the forward risk–neutral measure for maturity $T_1$ years (i.e. the EMM for numéraire $p(t,T_1)$). By decomposing the convertible bond into its option and bond parts, show that

$$X_0 = cp(0,T_1)\mathbb{E}_{\hat{\mathbb{Q}}}\left[(\hat{S}_{T_0} - \tfrac{1}{c})^+\right] + p(0,T_1)$$

where $\hat{S}_t = \frac{S_t}{p(t,T_1)}$.

10.3 The Ho–Lee model is an affine term structure model, i.e. bond prices are of the form

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)}$$

By substituting this expression into the term structure PDE, show that

$$B(t,T) = T - t \qquad A(t,T) = \int_t^T \theta(u)(u-T)\,du + \frac{1}{6}\sigma_r^2(T-t)^3$$

10.4 Show that

$$\frac{dp(t,T)}{p(t,T)} = r(t)\,dt - \sigma_r(T-t)\,dW_t$$

10.5 Fit the Ho–Lee model to the initial term structure of forward rates: Show that

$$\theta(t) = \sigma_r^2 t$$

10.6 Hence show that

$$r(t) = r_0 + \frac{1}{2}\sigma_r^2 t^2 + \sigma_r W_t$$

10.7 Hence, using the known initial bond prices and short rate dynamics, show that future bond prices are given by

$$p(t,T) = e^{-\frac{1}{2}\sigma_r^2 t(T-t)^2 - r(t)(T-t)}$$

What is the distribution of $p(t,T)$ under $\mathbb{Q}$?

10.8 Show that $\hat{S}_t$ has dynamics

$$\frac{d\hat{S}_t}{\hat{S}_t} = (\sigma_S + \sigma_r(T_1 - t))\,d\hat{W}_t$$

under the measure $\hat{\mathbb{Q}}$, where $\hat{W}_t$ is a $\hat{\mathbb{Q}}$–Brownian motion.
What is the distribution of $\hat{S}_t$ under $\hat{\mathbb{Q}}$?

10.9 Deduce that

$$X_0 = cS_0N(d_+) - p(0,T_1)N(d_-) + p(0,T_1)$$

where

$$d_\pm = \frac{\ln \frac{cS_0}{p(0,T_1)} \pm \frac{1}{2}\Sigma^2 T_0}{\sqrt{\Sigma^2 T_0}} \qquad \Sigma^2 = \frac{1}{T_0}\int_0^{T_1} ||\sigma_S + \sigma_r(T_1 - s)||^2 \, ds$$

11. The aim of this problem is to calculate the price of an in–arrears caplet in the Ho–Lee model, where the short rate has riskneutral dynamics

$$dr_t = \theta(t)\, dt + \sigma\, dW_t$$

Here, $\sigma$ is a constant, and $W_t$ is a 1–dimensional Brownian motion under the risk neutral measure $\mathbb{Q}$. The caplet has payoff

$$0.5 \max\{L - R_c, 0\}$$

at expiry = 1 year, where $L$ is the 6–month spot LIBOR rate in 6 months' time, and $R_c$ is the cap rate. Use the following data:

| $P(0,T)$ | $e^{-r_0T}$ |
|---|---|
| $r_0$ | 10% |
| $R_c$ | 12% |
| $\sigma$ | 10% |

Here $P(0,T)$ is the default–free zero coupon bond with face value 1 and maturity $T$.

We proceed as follows: We first show that the caplet is equivalent to a portfolio of put options on zero coupon bonds. Then we recast the Ho–Lee model within the HJM framework in order to fit it to the observed (flat) term structure, and calculate the prices of zero coupon bonds. Finally, we calculate the prices of vanilla options on zero coupon bonds.

11.1 First show that a caplet can be regarded as a portfolio of 6–month put options on the 1-year zero:

$$\begin{aligned}\text{Caplet } =& (1 + R_c\Delta T) \text{ put options on } P(t,T_2) \text{ with strike } \frac{1}{1+R_c\Delta T}\\ &\text{and maturity } T_1\end{aligned}$$

where $T_1 = 0.5$, $\Delta T = 0.5$, and $T_2 = T_1 + \Delta T = 1$.

11.2 The Ho–Lee model is an *affine* short rate model, with bond prices of the form $P(t,T) = e^{A(t,T)-B(t,T)r_t}$. By substituting this form of $P(t,T)$ into the term structure PDE, show that

$$\begin{aligned} B(t,T) &= T - t \\ A(t,T) &= -\int_t^T \theta(u)(T-u)\, du + \frac{1}{6}\sigma^2(T-t)^3 \end{aligned}$$

11.3 In order to fit the short rate model to the observed term structure, we recast it in the HJM framework. You may use the facts about the HJM model which are stated on the formula sheet.

Using the relation between forward rates and zero coupon bond prices and the value of $B(t,T)$, show that the instantaneous forward rate $f(t,T)$ has a constant "volatility" $\sigma$, i.e. that the forward rate dynamics are

$$df(t,T) = \alpha(t,T)\,dt + \sigma\,dW_t$$

for some function $\alpha(t,T)$.

11.4 Use the HJM drift conditions to show that $\alpha(t,T) = \sigma^2(T-t)$

11.5 Hence show that

$$r_t = r_0 + \tfrac{1}{2}\sigma^2 t^2 + \sigma W_t$$

and conclude that

$$dr_t = \sigma^2 t\,dt + \sigma\,dW_t$$

11.6 Next, show that

$$A(t,T) = -\frac{1}{2}\sigma^2 t(T-t)^2$$

and thus that zero coupon bond prices are given by

$$P(t,T) = e^{-\frac{1}{2}\sigma^2 t(T-t)^2-(T-t)r_t}$$

11.7 Now that bond prices have been found, we will price bond options. Recall the general option formula stated in the Formula Sheet. Change the numeraire to the 6-month zero coupon bond $P(0,T_1)$. Let $\hat{P}_t = \frac{P(t,T_2)}{P(t,T_1)}$. Write down the dynamics of $\hat{P}_t$ under $T_1$–forward neutral measure $\mathbb{Q}_1$.

11.8 Show that $\hat{P}_{T_1}$ is lognormally distributed under $\mathbb{Q}_1$, and find its distribution.

11.9 Similarly, find the dynamics of $\check{P}_t = \frac{1}{\hat{P}_t}$ under the $T_2$–forward measure $\mathbb{Q}_2$. Show that $\check{P}_{T_2}$ is lognormally distributed under $\mathbb{Q}_2$.

11.10 Deduce the following formula for a call option on $P(t,T_2)$ with strike $K$ and maturity $T_1$:

$$C_0 = P(0,T_2)N(d_+) - KP(0,T_1)N(d_-)$$

Write down expressions for $d_\pm$.

11.11 Use put–call parity and the table of the normal distribution to find the price of the original caplet.